ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM.

BY THE SAME AUTHOR.

THE THEORY AND PRACTICE OF ABSOLUTE MEASUREMENTS IN ELECTRICITY AND MAGNETISM.

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IN

ELECTRICITY AND MAGNETISM

BY

ANDREW GRAY, M.A., F.R.S.E.

PROFESSOR OF PHYSICS IN THE UNIVERSITY COLLEGE OF NORTH WALES

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PREFACE.

THE following statement was prefixed to the first edition of the present work, published in 1884 :—

"This little book was originally intended to be mainly a reprint of some papers on the Measurement of Electric Currents and Potentials in Absolute Measure contributed to *Nature* during the winter of 1882-3 ; but as these were being reprinted, many alterations and additions suggested themselves, which it was thought would render the book more generally useful. Most of the additional matter is mentioned in the introductory chapter, but I may here refer to a sketch of the theory of alternating machines, and of methods of measurement available in such cases, contained in Chapter X., and to Chapter XII. on the Dimensions of Units, which I have thought it desirable to introduce.*

"The work has of course no pretensions to being a complete treatise on Electrical and Magnetic Measure-

* Chapters IX. and XI. of the present edition.

ments, but is rather designed to give as far as is possible within moderate limits a clear account of the system of absolute units of measurement now adopted, and of some methods and instruments by which the system can be applied in both theoretical and practical work.

"I am under great obligations to Sir WILLIAM THOMSON and to Mr. J. T. BOTTOMLEY, who have kindly examined some of the proofs, and favoured me with valuable suggestions."

About a year ago the first volume of what is designed to be a more comprehensive treatise on this subject was published by Messrs. Macmillan and Co., under the title of *The Theory and Practice of Absolute Measurements in Electricity and Magnetism*, and the second volume is now in progress. It was intended that that work should supersede the original small book, which had been long out of print. It has, however, been represented to me that the smaller work has been found useful by students, and that a desire exists for a second edition on the original plan. I have therefore prepared the present volume, incorporating in it a few parts of the larger work, in which the original matter had been amplified and brought more nearly to date. I have besides revised the

whole, and made several additions which I hope may be found to add to its interest and usefulness. Among these are a fuller account of the Determination of *H*, a description of Sir William Thomson's Standard Electrical Instruments, a more complete treatment of the Graduation of Instruments, especially by Electrolysis, an extension of the Theory of Alternating Machines so as to include Dr. Hopkinson's Theory of the Working of Alternators in Series and in Parallel, additional information regarding the Measurement of Activity, &c., in the circuits of Alternators, and what I hope is an improved Chapter on the Theory of Dimensions of Physical Quantities. In the last mentioned part I have adopted Professor Rücker's suggestion that the dimensions of the electric and magnetic inductive capacities should be left undetermined, and regarded as so related as to render the dimensions of every physical quantity the same in the electrostatic as in the electromagnetic system of units.

Except in the part relating to Sir WILLIAM THOMSON's instruments, which has been revised by Sir WILLIAM himself and his assistant Mr. J. RENNIE, I have had little or no assistance in reading the proof sheets; and it is probable therefore that many slips and

inaccuracies have escaped my notice. I shall feel greatly obliged if any reader who may detect errors will kindly communicate them either to Messrs. Macmillan or to myself.

In conclusion I wish here to acknowledge with great respect the many and deep obligations I am under to Sir WILLIAM THOMSON, at whose suggestion I undertook the original work, and to whose encouragement and kindly help is due what little progress in scientific studies I have been able to make.

A. GRAY.

THE UNIVERSITY COLLEGE OF
NORTH WALES, BANGOR.
October, 1889.

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ERRATA.

P. 29, line 2 from top, *for* 64 *read* 16.
P. 29, line 8 from top, *for* $\frac{3}{64}$ *read* $\frac{3}{16}$.
P. 29, line 10 from top, *for* $\frac{3}{32}$ *read* $\frac{3}{16}$.

ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM.

CHAPTER I.

INTRODUCTORY.

TWENTY-FIVE years ago the experimental sciences of electricity and magnetism were in great measure mere collections of qualitative results, and, in a less degree, of results quantitatively estimated by means of units which were altogether arbitrary. These units, depending as they did on constants of instruments and conditions of experimenting which could never be made fully known to the scientific public, were a source of much perplexity and labour to every investigator, and to a great extent prevented the results which they expressed from bearing fruit to the furtherance of scientific progress. Now happily all this has been changed. The absolute system of units introduced by Gauss and Weber, and rendered a practical reality in this country by the labours of the British Association Committee on Electrical Standards, has changed experimental electricity and magnetism into sciences of which the very

essence is the most delicate and exact measurement, and enables their results to be expressed in units which are altogether independent of the instruments, the surroundings, and the locality of the investigator.

The record of the determinations of units made by members of the Committee, for the most part by methods and instruments which they themselves invented, forms one of the most interesting and instructive books * in the literature of electricity, and when the history of electrical discovery is written the story of their work will form one of its most important chapters. But besides placing on a sure foundation the system of absolute units they conferred a hardly less important benefit on electricians by giving them a convenient nomenclature for electrical quantities. The great utility of the practical units and nomenclature, which the Committee recommended, soon became manifest to every one who had to perform electrical measurements, and has led within the last few years to their adoption, with only slight alterations, by nearly all civilized nations. Although it is only five years since the recommendations † of the Paris Congress of Electricians were issued, they have been almost universally adopted and appreciated by those engaged in electrical work, and have thus begun to yield excellent fruit by rendering immediately available for comparison and as a basis for further research the results of experimenters in all parts of the world.

But in order that the full benefit of the conclusions of the Paris Congress may be obtained it is essential in the first place that convenient instruments should be

* Reports of the British Association Committee on Electrical Standards, edited by Prof. Jenkin, F.R.S.

† See Appendix below.

used, adapted to give directly, or by an easy reduction from their indications, the number of amperes of current flowing in a particular circuit, and the number of volts of difference of potential between any two points in that circuit. To be generally useful in practice these instruments should be easily portable, and should have a very large range of sensibility; so that, for example, the instrument, which suffices to measure the full potential produced by a large dynamo-electric machine, may be also available for testing, if need be, the resistances of the various parts of the armature and magnets by the readiest and most satisfactory method; namely, by comparing by means of a galvanometer of high resistance the difference of potential between the two ends of the unknown resistance with that between the ends of a known resistance joined up in the same electrical circuit. In like manner the ampere measurer should be one that could be introduced without sensible disturbance into a circuit of low resistance to measure either a small fraction of an ampere, or the whole current flowing through a circuit containing a large number of electric lamps. These conditions are more or less fulfilled by a large variety of practical instruments recently patented by different inventors. Among these is a very complete set of potential and current measurers, due to Sir William Thomson, and adapted for all kinds of work. My main purpose in the present work is to give an account, both theoretical and practical, of the measurement of currents and potentials in absolute measure; and to apply this to the graduation of instruments for use in practical electrical work. Of such instruments I shall take some of Sir William Thomson's as examples, and after describing them, show how they may be graduated, or their graduation tested,

by the experimenter himself. It will be convenient to introduce definitions of absolute units of measurement of magnetic and electric quantities when they are required; and in so doing I shall endeavour to give a clear account of the foundation of the absolute system of electrical measurement, and to show how from the fundamental units are derived the practical units of current, quantity, potential, and resistance. I shall then give and explain a few rules for the calculation of currents and resistances in derived circuits: and describe among others some methods of determining resistances which are useful in many important practical cases, but which so far as I know are not treated of in the ordinary text-books of electricity. The remainder of the work will contain a brief account of the measurement of energy spent in the circuits of generators transmitting power to electric lamps or motors, or in the charging circuit of a secondary battery; a chapter on the practical determination in absolute units of the intensities of powerful magnetic fields, such as those of the field magnets of dynamo machines, and another on the dimensions of units; and, lastly, a few tables of useful electrical data.

CHAPTER II.

DETERMINATION OF THE HORIZONTAL COMPONENT OF THE EARTH'S MAGNETIC FIELD.

ALL the methods by which galvanometers may be graduated so as to measure currents and potentials in absolute units, involve, directly or indirectly, a comparison of the indications of the instrument to be graduated with those of a standard instrument, of which the constants are fully known for the place at which the comparison is made. There are various forms of such standard instruments, as, for example, the tangent galvanometer which Joule made, consisting of a single coil of large radius and a small needle hung at its centre, or the Helmholtz modification of the same instrument with two large equal coils placed side by side at a distance apart equal to the radius of either; or some form of "dynamometer," or instrument which instead of the needle of the galvanometer has a movable coil, in which the whole or a known fraction of the current in the fixed coil flows. The measurement consists essentially in determining the couple which must be exerted by the earth's magnetic force on the needle or suspended coil, in order to equilibrate that exerted by the current. But the former depends on the value, usually denoted by H, of the horizontal component of the earth's magnetic force, and it is necessary therefore, except when some such method as that of Kohlrausch,

referred to below, is used, to know the value of that quantity in absolute units.

The value of H may be determined in various ways, and I shall here content myself with describing the method which is most convenient in practice. It consists in finding (1) the angle through which the needle of a magnetometer is deflected by a magnet placed in a given position at a given distance, (2) the period of vibration of the magnet when suspended horizontally in the earth's field, so as to be free to turn round a vertical axis. The first operation gives an equation involving the ratio of the magnetic moment of the magnet to the horizontal component H of the terrestrial magnetic force, the second an equation involving the product of the same two quantities. I shall describe this method, which was given by Gauss,* somewhat in detail.

A very convenient form of magnetometer is that devised by Mr. J. T. Bottomley, and made by hanging within a closed chamber, by a silk fibre from 6 to 10 cms. long, one of the little mirrors with attached magnets used in Thomson's reflecting galvanometers. The fibre is carefully attached to the back of the mirror, so that the magnets hang horizontally and the front of the mirror is vertical. The closed chamber for the fibre and mirror is very readily made by cutting a narrow groove to within a short distance of each end, along a piece of mahogany about 10 cms. long. This groove is widened at one end to a circular space a little greater in diameter than the diameter of the mirror. The piece of wood is then fixed with that end down in a horizontal base-piece of wood furnished with three levelling screws. The groove is thus

* "Intensitas vis magneticae ad mensuram absolutam revocata."—*Comment. Soc. Reg. Gotting.* 1833.

placed vertical; and the fibre carrying the mirror is suspended within it by passing the free end of the fibre through a small hole at the upper end of the groove, adjusting the length so that the mirror hangs within the circular space at the bottom, and fixing the fibre at the top with wax. When this has been done, the chamber is closed by covering the face of the piece of wood with a strip of glass, which may be kept in its place either by cement, or by proper fastenings which hold it tightly against the wood. By making the distance between the back and front of the circular space small, and its diameter very little greater than that of the mirror, the instrument can be made very nearly "dead beat,"—that is to say, the needle when deflected through any angle comes to rest at once, almost without oscillation about its position of equilibrium. A magnetometer can be thus constructed at a trifling cost, and it is much more accurate and convenient than the magnetometers furnished with long magnets frequently used for the determination of *H*; and as the poles of the needle may always in practice be taken at the centre of the mirror, the calculations of results are much simplified.

The instrument is set up with its glass front in the magnetic meridian, and levelled so that the mirror hangs freely inside its chamber. The foot of one of the levelling screws should rest in a small conical hollow cut in the table or platform, of another in a V-groove the axis of which is in line with the hollow, and the third on the plane surface of the table or platform. When thus set up the instrument is perfectly steady, and if disturbed can in an instant be replaced in exactly the same position. A beam of light passes through a slit, in which a thin vertical cross-wire is fixed, from a lamp placed in front of

the magnetometer, and is reflected, as in Thomson's reflecting galvanometer, from the mirror to a scale attached to the lamp-stand, and facing the mirror. The lamp and scale are moved nearer to or farther from the mirror, until the position at which the image of the cross-wire of the slit is most distinct is obtained. It is convenient to make the horizontal distance of the mirror from the scale for this position if possible one metre. The lamp-stand should also have three levelling screws, for which the arrangement of conical hollow, V-groove, and plane should be adopted. The scale should be straight, and placed with its length in the magnetic north and south line; and the lamp should be so placed that the incident and reflected rays of light are in an east and west vertical plane, and that the spot of light falls near the middle of the scale. To avoid errors due to variations of length in the scale, it should be glued to the wooden backing which carries it, not simply fastened with drawing pins as is often the case.

The magnetometer having been thus set up, four or five magnets, each about 10 cms. long and ·1 cm. thick, and tempered glass-hard, are made from steel wire. This is best done as follows. From ten to twenty pieces of steel wire, each perfectly straight and having its ends carefully filed so that they are at right angles to its length, are prepared. These are tied tightly into a bundle with a binding of iron wire and heated to redness in a bright fire. The bundle is then quickly removed from the fire, and plunged with its length vertical into cold water. The wires are thus tempered glass-hard without being seriously warped. They are then magnetized to saturation in a helix by a strong current of electricity. A horizontal east and west line passing through the mirror is now

laid down on a convenient platform (made of wood put together without iron and extending on both sides of the magnetometer) by drawing a line through that point at right angles to the direction in which a long thin magnet hung by a single silk fibre there places itself (see also p. 23). One of the magnets is placed, as shown in Fig. 1, with its length in that line, and at such a distance that a convenient deflection of the needle is produced. This deflection is noted and the deflecting magnet turned end for end, and the deflection again noted. In the same way a pair of observations are made with the magnet at the same distance on the opposite side of the magnetometer; and the mean of all the observations is taken. These deflections from zero ought to be as nearly as may be

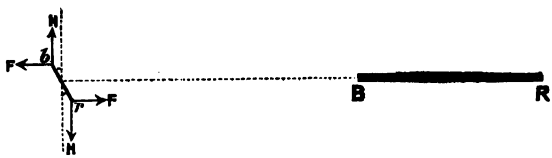

FIG. 1.

the same, and if the magnet is properly placed, they will exactly agree; but the effect of a slight error in placing the magnet will be nearly eliminated by taking the mean deflection. The distance in cms. between the two positions of the centre of the magnet is also noted, and is taken as twice the distance of the centre of the magnet from that of the needle. The same operation is gone through for each of the magnets, which are carefully kept apart from one another during the experiments. The results of each of these experiments give an equation involving the ratio of the magnetic moment of the magnet to the value of *H*. Thus if

M denote the magnetic moment of the magnet, M' the magnetic moment of the needle, r the distance of the centre of the magnet from the centre of the needle, 2λ the distance between the "poles" (p. 36) of the magnet which, for a nearly uniformly magnetized magnet of the dimensions stated above, is nearly equal to its length, and $2\lambda'$ the distance between the poles of the needle, r, λ, and λ' being all measured in cms., we have for the repulsive force (denoted by F in Fig. 1) exerted on the blue* pole of the needle by the blue pole of the magnet, supposed nearest to the needle, as in Fig. 1, the value $\frac{M}{2\lambda} \cdot \frac{M'}{2\lambda'} \cdot \frac{1}{(r - \lambda)^2}$, since the value of λ' is small compared with λ. Similarly for the attraction exerted on the same pole of the needle by the red pole of the magnet, we have the expression $\frac{M}{2\lambda} \cdot \frac{M'}{2\lambda'} \cdot \frac{1}{(r + \lambda)^2}$. Hence the total repulsive force exerted by the magnet on the blue pole of the needle is

$$\frac{MM'}{4\lambda\lambda'} \left\{ \frac{1}{(r - \lambda)^2} - \frac{1}{(r + \lambda)^2} \right\} \text{ or } M \frac{M'}{\lambda'} \cdot \frac{r}{(r^2 - \lambda^2)^2}.$$

Proceeding in a precisely similar manner, we find that the magnet of moment M exerts an attractive force equal to $M \frac{M'}{\lambda'} \cdot \frac{r}{(r^2 - \lambda^2)^2}$ on the red pole of the magnet. The needle is therefore acted on by a "couple" which tends to turn it round the suspending fibre as an axis, and the amount of this couple, when the angle of deflection is θ, is plainly equal to $MM' \frac{2r}{(r^2 - \lambda)^2} \cos \theta$. But for equilibrium this

* The convention according to which magnetic polarity of the same kind as that of the earth's northern regions is called blue, and magnetic polarity of the same kind as that of the earth's southern regions is called red, is here adopted. The letters B, R, *b*, *r*, in the diagrams denote blue and red.

couple must be balanced by $M'H \sin\theta$; hence we have the equation—

$$\frac{M}{H} = \frac{(r^2 - \lambda^2)^2}{2r} \tan\theta \quad . \quad . \quad . \quad . \quad (1)$$

If the arrangement of magnetometer and straight scale described above is adopted, the value of $\tan\theta$ is easily obtained, for the number of divisions of the scale which measures the deflection, divided by the number of such divisions in the distance of the scale from the mirror, is then equal to $\tan 2\theta$.

Instead of in the east and west horizontal line through the centre of the needle, the magnet may be placed, as represented in Fig. 2, with its length east and west, and its centre in the horizontal north and south line through the centre of the needle. If we take M, M', λ, λ', and r to have the same meaning as before, we have, for the distance of either pole of the magnet from the needle, the expression $\sqrt{r^2 + \lambda^2}$. Let us consider the force acting on one pole, say the red pole of the needle. The red pole of the magnet exerts on it a repulsive force, and the blue pole an attractive force. Each of these forces has the value $\frac{M}{2\lambda} \cdot \frac{M'}{2\lambda'} \cdot \frac{1}{r^2 + \lambda^2}$. But the diagram shows that they are equivalent to a single force, F, in a line parallel to the magnet, tending to pull the red pole of the needle towards the left. The magnitude of this resultant force is plainly $2\,\frac{M}{2\lambda} \cdot \frac{M'}{2\lambda'} \cdot \frac{\lambda}{(r^2 + \lambda^2)^{\frac{3}{2}}}$ or $\frac{MM'}{2\lambda'(r^2 + \lambda^2)^{\frac{3}{2}}}$. In the same way it can be shown that the action of the magnet on the red pole of the needle is a force of the same amount tending to pull the blue pole of the needle towards the right. The needle is, therefore, subject to no force tending to

produce motion of translation, but simply to a "couple" tending to produce rotation. The magnitude of this couple when the needle has been turned through an angle θ, is $\frac{MM'}{2\lambda'} \cdot \frac{2\lambda' \cos \theta}{(r^2 + \lambda^2)^{\frac{3}{2}}}$, or $\frac{MM'}{(r^2 + \lambda^2)^{\frac{3}{2}}} \cos \theta$. If there be

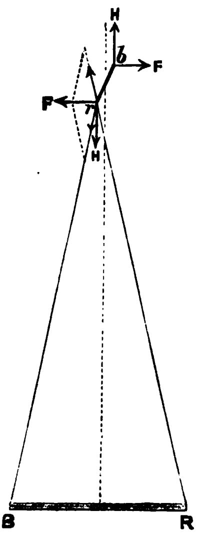

FIG. 2.

equilibrium for the deflection θ, this couple must be balanced by that due to the earth's horizontal force, which, as before, has the value $M'H \sin \theta$. Hence equating these two couples we have—

$$\frac{M}{H} = (r^2 + \lambda^2)^{\frac{3}{2}} \tan \theta \quad . \quad . \quad . \quad . \quad (2)$$

Still another position of the deflecting magnet relatively to the needle may be found a convenient one to adopt. The magnet may be placed still in the east and west line, but with its centre vertically above the centre of the needle. The couple in this case also is given by the formula just found, in which the symbols have the same meaning as before.

The greatest care should be taken in all these experiments, as well as in those which follow, to make sure that there is no movable iron in the vicinity, and the instruments and magnets should be kept at a distance from any iron nails or bolts there may be in the tables on which they are placed.

We come now to the second operation, the determination of the period of oscillation of the deflecting magnet when under the influence of the earth's horizontal force alone. The magnet is hung in a horizontal position in a double loop formed at the lower end of a single fibre of unspun silk, attached by its upper end to the roof of a closed chamber. A box about 30 cms. high and 15 cms. wide, having one pair of opposite sides, the bottom, and the roof made of wood, and the remaining two sides made of plates of glass, one of which can be slided out to give access to the inside of the chamber, answers very well. The fibre may be attached at the top to a horizontal axis which can be turned round from the outside so as to wind up or let down the fibre when necessary. The suspension-fibre is so placed that two vertical scratches, made along the glass sides of the box, are in the same plane with the magnet when the magnet is placed in its sling, and the box is turned round until the magnet is at right angles to the glass sides. A paper screen with a small hole in it is then set up at a little distance in such a position that the

hole is in line with the magnet, and therefore in the same plane as the scratches. The magnetometer should be removed from its stand and this box and suspended needle put in its place. If the magnet be now deflected from its position of equilibrium and then allowed to vibrate round a vertical axis, it will be seen through the small hole to pass and re-pass the nearer scratch, and an observer keeping his eye in the same plane as the scratches can easily tell without sensible error the instant when the magnet passes through the position of equilibrium. Or, a line may be drawn across the bottom of the box so as to join the two scratches, and the observer keeping his eye above the magnet and in the plane of the scratches may note the instant when the magnet, going in the proper direction, is just parallel to the horizontal line. The operator should deflect the magnet by bringing a small magnet near to it, taking care to keep this small deflecting magnet always as nearly as may be with its length in an east and west line passing through the centre of the suspended magnet. If this precaution be neglected the magnet may acquire a pendulum motion about the point of suspension, which will interfere with the vibratory motion in the horizontal plane. When the magnet has been properly deflected and left to itself, its range of motion should be allowed to diminish to about 3° on either side of the position of equilibrium before observation of its period is begun. When the amplitude has become sufficiently small, the person observing the magnet says sharply the word "Now," when the nearer pole of the magnet is seen to pass the plane of the scratches in either direction, and another observer notes the time on a watch having a seconds hand. With a good watch having a centre seconds hand moving round a dial divided into quarter-seconds,

the instant of time can be determined with greater accuracy in this way than by means of any of the usual appliances for starting and stopping watches, or for registering on a dial the position of a seconds hand when a spring is pressed by the observer. The person observing the magnet again calls out "Now" when the magnet has just made ten complete to and fro vibrations, again after twenty complete vibrations, and, if the amplitude of vibration has not become too small, again after thirty; and the other observer at each instant notes the time by the watch. By a complete vibration is here meant the motion of the magnet from the instant when it passes through the position of equilibrium in either direction, until it next passes through the position of equilibrium going in the same direction. The observers then change places and repeat the same operations. In this way a very near approach to the true period is obtained by taking the mean of the results of a sufficient number of observations, and from this the value of the product of m and H can be calculated.

For a small angular deflection θ of the vibrating magnet from the position of equilibrium the equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{MH}{\mu}\theta = 0,$$

where μ is the moment of inertia of the vibrating magnet round an axis through its centre at right angles to its length. The solution of this equation is

$$\theta = A \sin\left\{ \sqrt{\frac{MH}{\mu}}\, t - B \right\}$$

and therefore for the period of oscillation T we have

$$T = 2\pi \sqrt{\frac{\mu}{MH}}.$$

Hence we have

$$MH = \frac{4\pi^2\mu}{T^2}$$

Now, since the thickness of the magnet is small compared with its length, if W be the mass of the magnet and $2l$ its actual length, μ is $Wl^2/3$, and therefore

$$MH = \frac{4\pi^2 l^2 W}{3T^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining this with the equation (1) already found we get for the arrangement shown in Fig. 1,

$$M^2 = \frac{2}{3} \cdot \frac{\pi^2(r^2 - \lambda^2)^2 l^2 W \tan\theta}{T^2 r} \quad . \quad . \quad (4)$$

and

$$H^2 = \frac{8}{3} \cdot \frac{\pi^2 l^2 r W}{T^2(r^2 - \lambda^2)^2 \tan\theta} \quad . \quad . \quad . \quad (5)$$

If either of the other two arrangements be chosen we have from equations (2) and (3)

$$M^2 = \frac{4}{3} \cdot \frac{\pi^2 l^2}{T^2}(r^2 + \lambda^2)^{\frac{3}{2}} \, W \tan\theta \quad . \quad . \quad (6)$$

and

$$H^2 = \frac{4}{3} \cdot \frac{\pi^2 l^2 W}{(r^2 + \lambda^2)^{\frac{3}{2}} \, T^2 \tan\theta} \quad . \quad . \quad . \quad (7)$$

Various corrections which are not here made are of course necessary in a very exact determination of H. The virtual length of the magnet—that is, the distance between its poles or "centres of 'gravity' of magnetic polarity" as they have been called *—should be determined by experiment. This may be done by observing the deflections θ

* Strictly speaking no such points exist. See p. 36 below.

and θ' of the magnetometer needle produced by the magnet when placed in the position shown in Fig. 1 at distances r and r', both great in comparison with l, from the centre of the needle. We have the equations

$$\frac{M}{H} = \frac{(r^2 - \lambda^2)^2}{2r} \tan\theta = \frac{(r'^2 - \lambda^2)^2}{2r'} \tan\theta'$$

and therefore,

$$\lambda^2 = \frac{r'^2\sqrt{r\tan\theta'} - r^2\sqrt{r'\tan\theta}}{\sqrt{r\tan\theta'} - \sqrt{r'\tan\theta}} \quad . \quad . \quad (8)$$

Preferable methods of taking account of the magnetic distribution will be found at pp. 28, 29 below. Allowances should be made for the magnitude of the arc of vibration; the torsional rigidity of the suspension fibre of the magnetometer in the deflection experiments (pp. 9—12), and of the suspension fibre of the magnet in the oscillation experiments; the frictional resistance of the air to the motion of the magnet; the virtual increase of inertia of the magnet due to motion of the air in the chamber; and the effect of induction and, if necessary, of changes of temperature in producing temporary changes in the moment of the magnet. The correction for an arc of oscillation of 6° is a diminution of the observed value of T of only $\frac{1}{60}$ per cent., and for an arc of 10° of $\frac{1}{20}$ per cent. Of the other corrections that for induction is no doubt the most important; but its amount for a magnet of glass-hard steel, nearly saturated with magnetism, and in a field so feeble as that of the earth, may, if only a roughly accurate result is required, be neglected.

This correction arises from the fact that the magnet in the deflection experiments is placed in the magnetic east and west line, whereas in the oscillation experiments it is

placed north and south, and is therefore subject in the latter case to an increase of longitudinal magnetization from the action of terrestrial magnetic force. The increase of magnetic moment may be determined by the following method, which is due to Mr. Thomas Gray. Place the magnet within, and near the centre of, a helix, considerably longer than the magnet and made of insulated copper wire. Place the helix and magnet in position either as shown in Fig. 2, or as in Fig. 3, for giving a deflection of the magnetometer needle, and read the deflection. Then pass such a current through the wire of the helix as will give by electromagnetic induction a magnetic field within the helix nearly equal to the horizontal component of the earth's field, and again observe the deflection of the magnetometer needle. The intensity of the field thus produced within the helix at points not near the ends is given in C.G.S. units (p. 43) by the formula $4\pi nC$, where n is the number of turns per centimetre of length of the helix, and C is the strength of the current in C.G.S. units (pp. 44—48). Experiments may be made with different strengths of current, and the results put down in a short curve, from which the correction can be at once read off when the approximate field has been determined by the method of deflection and oscillation described above. Care must of course be taken in experimenting to eliminate the deflection of the magnetometer needle caused by the current in the coil. This is easily done by observing the deflection produced by the current when the magnet is not inside the coil and subtracting this from the previous deflection, or by arranging a compensating coil through which the same current passes. This plan has several advantages (see p. 31 below). The change of magnetic moment produced

in hard steel bars, the length of which is 12 cms. and diameter ·2 cm., and previously magnetized to saturation, is, according to Mr. Thomas Gray's experiments, about $\frac{1}{20}$ per cent. (For particulars of actual experiments, see pp. 30—36 below.)

The deflection experiments are, as stated above, to be performed with several magnets, and when the period of oscillation of each of these has been determined, the magnetometer should be replaced on its stand, and the deflection experiments repeated, to make sure that the magnets have not changed in strength in the meantime. The length of each magnet is then to be accurately determined in centimetres, and its weight in grammes; and from these data and the results of the experiments the values of M and of H can be found for each magnet by the formulas investigated above. Equation (5) is to be used in the calculation of H when the arrangement of magnetometer and deflecting magnet, shown in Fig. 1, is adopted, equation (7), when that shown in Fig. 2 is adopted.

The object of performing the experiments with several magnets, is to eliminate as far as possible errors in the determination of weight and length. The mean of the values of H, found for the several magnets, is to be taken as the value of H at the place of the magnetometer. We shall show in the next chapter how to apply this value to the measurement of currents.

The following is an account of a determination of H made by this method, with several improvements in the practical carrying of it out, by Mr. Thomas Gray in the Physical Laboratory of the University of Glasgow, during the summer of 1885. The apparatus and its arrangement is shown in Fig. 3. T represents a table which supports

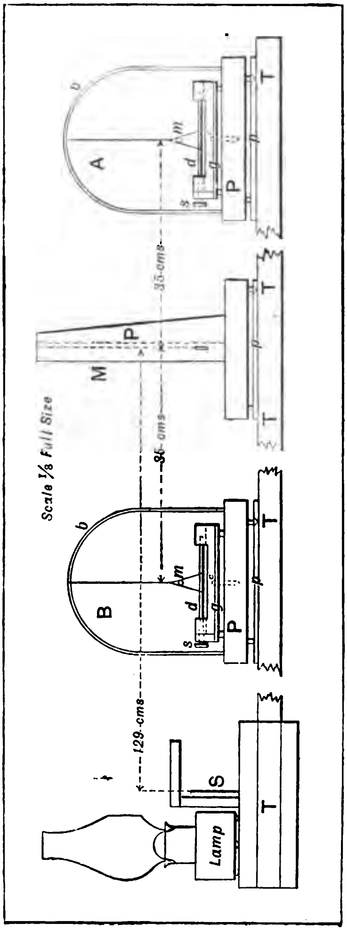

Fig. 3.

the magnetometer M, two stands A and B for the deflecting magnets, and a lamp and scale S. The magnetometer consisted, as described above, of a light mirror about ·8 cm. in diameter, suspended by a single silk fibre within a recess in a block of wood, and carrying on its back two magnets each 1 cm. long and ·08 cm. in diameter. Two holes cut in the wood at right angles to one another (and plugged when not in use), permitted the position of the mirror and magnets to be seen and adjusted. [A preferable form of magnetometer since adopted consists of a mirror and attached magnet suspended within a glass-tube from a brass mounting at the upper end which allows the fibre to be wound up or down. For definiteness and ease of determination of the magnetic centre of the needle, a single small cylindrical magnet is used, carried at the lower end by a short strip of aluminium to which the mirror is attached. At the lower end half the tube is cut away over a length of two or three cms., and the part remaining closes the back of the chamber in which the mirror hangs. The sides of the chamber are of wood attached to the base piece, and the front, or side toward the lamp, is closed by a sliding panel of glass which can be placed in either of two slots so as to give a greater or less space for the mirror to swing in.] The sole plate P, made of mahogany, is supported on three brass feet, which rest in a hole, slot, and plane arrangement cut as described above in a horizontal plate of glass cemented to the table.

The deflector stands A, B rest each on a base plate P, of mahogany, supported, according to the hole, slot, and plane device, in precisely the same way as the magnetometer, on plates of glass p, p, cemented to the table T. Each stand consists of a horizontal carriage for the deflec-

tor magnet, and is constructed as follows: A strip of hard wood, about 13 cms. long and 4 cms. broad, has a **V**-shaped groove run along its whole length in the middle of one side. One end is faced with a plate of brass in which a brass screw works, and the piece is cemented with the groove upwards to a plate of glass *g*. This plate is supported on three feet of hard wood, resting on the mahogany sole plate *P*, and is free to turn in azimuth round a closely fitting centre pivot *c* fixed in the sole plate. The apparatus is so adjusted that the bottom of the **V**-groove is just over the pivot *c*. The magnet when placed in the carriage lies along the groove, and the screw *s* serves to give a fine adjustment of the position of one end which abuts against it. Over each carriage a wire of brass or copper bent into a semi-circle serves as a support for a suspension fibre with double loop, by which the deflector can be suspended for purposes of adjustment or for the oscillation experiments. A glass shade can be placed on the plate *P* to prevent currents of air from disturbing the magnet in the oscillation experiments.

In Fig. 3 the deflecting magnets *d*, *d*, are shown in positions at equal distances east and west of the magnetometer, at a distance of 70 cms. between their centres. Four plates of glass are fixed to the table in two end on positions and in two side on positions, each pair of positions being at equal distances from the magnetometer needle, and on opposite sides of it. The scale *S*, shown at a distance from the mirror of 129 cms., is a millimetre scale carefully divided on transparent glass so that the spot of light may be observed either from the front or the back.

The first adjustment, made in setting up the apparatus, was to place the table so that the line joining the centres

of A and B should be exactly at right angles to the magnetic meridian. This was done by one or other of the following two methods according as (*a*) the end-on, or (*b*) the side-on position was required. (*a*) After the adjustment had been first roughly made, a plane circuit was formed by stretching a thin wire along the line joining the centres of A, B under the magnetometer needle, and then carrying the wire back, either above the magnetometer, or below it, at a greater distance, in a vertical plane. An electric current was then sent through the wire, and the table T, with the apparatus turned until the current produced no deflection of the needle. (*b*) One of the deflecting magnets was placed in its carriage, either south or north of the needle, and lifted out of the V-groove by the suspension fibre, and the table turned until the suspended magnet produced no deflection of the magnetometer needle. The magnet and needle were then in one line, and if the needle was in its proper position this line produced through the centre of the needle passed through the position of the deflector on the other side. The deflector was placed on the opposite side of the needle, and the table T, turned until no deflection was obtained. The position of the needle was then altered, if necessary, by the levelling screws until the positions of the table for no deflection with the magnet first on one side then on the other of the magnetometer were coincident. If this could not be done the plates p were not placed with sufficient accuracy, and their position had to be changed. This process gave the direction of the magnetic meridian with accuracy and ensured that the plates p in the north and south line were properly placed on the table. The two methods taken together ensured that all four plates p were properly placed.

Deflectors of different relative lengths and thicknesses, and of different degrees of hardness, were used. These were originally magnetized by placing them between the poles of a large Ruhmkorff magnet excited by a considerable current, and afterwards by the same magnet excited by a much stronger current. The relative strengths of the magnets were unchanged by the second magnetization, and their absolute strengths only very slightly. The dimensions are given in the table of results, p. 35 below. The method of observing the deflections was as follows: According to a suggestion of Sir William Thomson two deflectors were used at the same time, one on each side of the magnetometer. This arrangement was more symmetrical than that of a single deflector, and, what was of very great importance, it enabled a readable deflection to be obtained with the magnets at a much greater distance from the needle, thus diminishing error due to uncertainty as to the actual magnetic distribution. As each magnet was transferred on its carriage from one glass plate to another the magnet was not handled during the experiments. One deflector *A* was placed east another *B* west of the magnetometer, and the plate *g* turned for each until their lengths were accurately in the east and west line, and their poles so turned that each magnet gave a deflection of the needle to the same side of zero; and the deflection was then noted. The plates *g* were then turned through 180°, and the deflection on the opposite side of zero read off. The carriages were then turned back to the first position and the deflection again read. The difference between the mean of the first and third readings and the second reading gave twice the deflection for the position of the magnets. The same operation was then repeated with the deflector in inter-changed positions. Two similar

series of observations were next made with the magnets in the north and south line through the magnetometer and at equal distances on opposite sides of the needle. The mean deflection for the east and west positions and that for the north and south positions were calculated, and the results were used in the calculation of *H* in the manner described below.

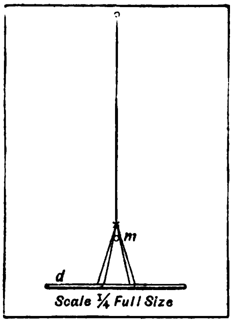

FIG. 4.

After the deflection observations for a particular magnet had been completed, the magnetometer was removed and the deflector stand put in its place. The magnet was suspended from the brass bow *b* over its carriage by a length of single cocoon fibre, in a double stirrup formed by twice doubling the lower end of the fibre and knotting. The suspension thus obtained was sufficiently fine to be practically devoid of inertia, and long enough to give a negligible moment of torsion. The magnet was deflected in the manner already described (p. 14 above), and then

left to oscillate. The period was observed in some cases by noting the times of the successive transits of the needle across the vertical cross wire of an observation telescope; but the method finally adopted was to attach to the stirrup as shown in Fig. 4 a light silvered mirror *m* (·3 cm. in diameter and ·01 gramme in mass), and to use the same lamp and scale as in the deflection experiments. This latter arrangement enabled the amplitude of oscillation to be reduced to less than a degree and so reduced to zero the correction necessary for arc. The moment of inertia of the mirror was only about $\frac{1}{40000}$ of that of the deflector, and its neglect therefore introduced an error of only $\frac{1}{400}$ per cent.

Time was observed in these experiments by means of a very accurate watch provided with a centre seconds hand moving round a dial divided into quarter seconds. When two observers were available, one counted the oscillations and called sharply "Now" at the end of every four or five periods, while the other observed the time at each call. When only one observer counted the oscillations he used a chronometer beating half seconds. Having read time, he counted the beats until he could observe a transit. He then counted the beats until he observed another transit. From the result he estimated the number of periods in one minute, and therefore observed the time of the first transit after each minute so long as there was sufficient amplitude. The fractions of half seconds were estimated from the positions of the magnet at the beat next before and the beat next after the transit. With the mirror and scale arrangement these observations could be made with great accuracy.

The observations were combined in the following manner so as to give the most probable value of the

period. Supposing the number of observations to have been even, $2n$ say. The interval between the nth observation and the $(n+1)$th, three times that between the $(n-1)$th and the $(n+2)$th, five times that between the $(n-2)$th and the $(n+3)$th, and so on to that between the 1st and the $2n$th were added together, the sum divided by the sum of the series $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$, and the result by the number of periods (which was the same in each case) between each successive pair of observations. This gave the average period to a high degree of approximation. If an odd number of observations $(2n+1)$ was taken, the interval between the nth and the $(n+2)$th, twice that between the $(n-1)$th and the $(n+3)$th, three times that between the $(n-2)$th and the $(n+4)$th, and so on to the 1st and $(2n+1)$th, were added together, the sum divided by twice the sum of the series $1^2 + 2^2 + 3^2 + \ldots + n^2$, and the result divided by the number of periods in each interval gave the average period. The period adopted was always the mean of those given by two closely agreeing sets of observations.

Assuming (see p. 36) that the magnet has two definite poles, that is (in this connection) points at which the whole of the free magnetism in each half of the magnet may be supposed concentrated in considering the external action of the magnet (an assumption not seriously erroneous in the case of the thin magnets and the distances used); the distance between them can be calculated from the results of deflection experiments in the side-on and end-on positions obtained as described above, since the effect of the distribution is opposite in the two cases. For if r be the distance for the end-on position, θ the deflection, and r', θ' the distance and deflection for the side-on

position, we have by equating the values of M/H given by equations (1) and (2):

$$\frac{(r'^2 - \lambda^2)^2}{2r(r'^2 + \lambda^2)^{\frac{3}{2}}} = \frac{\tan\theta}{\tan\theta'}.$$

Expanding the numerator and denominator of each side and neglecting terms smaller than those of the second order we get:

$$\lambda^2 = \frac{r^3\theta - 2r'^3\theta'}{2r\theta + 3r'\theta'} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

By this equation the value of λ used in the calculation of H and M was found. The results for magnets of different lengths and diameters are interesting in themselves.

The moment of inertia of the bar was found by weighing the bar and carefully measuring its length and cross-section, and calculating for a vertical axis through the centre of the magnet supposed hung horizontally. The axis of suspension of the magnet in any case was not, however, that vertical, but another near it owing to the compensation for the tendency of the magnet to dip in the earth's field. The distance between these two axes can be found approximately for each magnet from the magnetic moment, mass, and length as given in the table below, and is so small that any error caused by supposing the magnet simply to vibrate round the former vertical is well within the possible limit of accuracy.

For a cylindrical magnet of mass W, actual length $2l$ and diameter d, the moment of inertia is $W(l^2/3 + d^2/16)$. Hence (3) becomes:

$$MH = \frac{4}{3}\frac{\pi^2(l^2 + \frac{3}{16}d^2)W}{T^2} \quad . \quad . \quad . \quad (10)$$

Hence for a single deflector we get instead of equations

(4), (5), (6), (7) equations obtained from these by substituting instead of l^2, $l^2 + 3d^2/64$.

If two deflectors be used, each of the actual length $2l$, and diameter d, but of masses W_1, W_2, periods T_1, T_2, and nearly equal effective lengths which give a mean λ, we get from (1) and (2) instead of (5) and (6) for the end-on and side-on positions respectively:

$$H^2 = \frac{8}{3}\,\frac{\pi^2 r\,(l^2 + \frac{3}{64}d^2)\,(T_1^2 W_2 + T_2^2 W_1)}{(r^2 - \lambda^2)^2\, T_1^2 T_2^2 \tan\theta} \qquad (11)$$

$$H^2 = \frac{4}{3}\,\frac{\pi^2\,(l^2 + \frac{3}{32}d^2)\,(T_1^2 W_2 + T_2^2 W_1)}{(r'^2 + \lambda^2)^{\frac{3}{2}}\, T_1^2 T_2^2 \tan\theta}\,. \qquad (12)$$

In these formulas θ and θ' are the angular deflections found from the mean readings taken as described above (p. 24).

There are two corrections for alteration of moment of the magnet, produced (1) by variation of temperature, (2) by induction when the magnet is in or near the magnetic meridian when oscillating. The first correction was found by placing the magnet within a bath, in one of two principal positions at such a distance from the magnetometer needle that a deflection of 1,000 divisions was obtained, and then raising the temperature through about 40° C. It was found that such a rise of temperature produced a change of deflection of only about two divisions. Thus the magnets changed in magnetic moment by only $\frac{1}{200}$ per cent. for a change of temperature of 1° C. Hence as the variation of temperature in the experiments never exceeded 2° C. or 3° C. this correction was neglected.

The correction for induction was found by immersing the deflecting magnet in an artificially produced magnetic field of known strength, and ascertaining the alteration of

magnetic moment which resulted. The field was produced by surrounding the magnet with a magnetizing coil, and its intensity calculated from the number of turns of wire per unit of length of the coil and the current-strength, which was measured. The coil was sufficiently long to project beyond the magnet at each end some distance, so that the magnetic field was uniform, and equal to $4\pi nC$, where n = number of turns per cm. of length, and C the current strength in C.G.S. units (see below, Chap. III.).

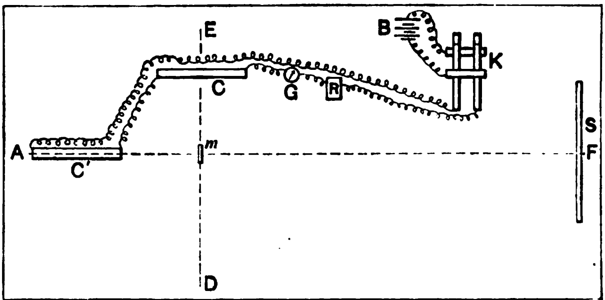

FIG. 5.

Fig. 5 shows the arrangement of apparatus for these experiments; m is the magnetometer needle, C, C' are coils each consisting of silk-covered copper wire wound on glass tubes 5 cm. in external diameter, S is the lamp scale, R a box of resistance coils, G the current galvanometer, K a reversing key, and B a battery. DE represents a horizontal line through the needle and in the magnetic meridian, and AF a horizontal line at right angles to DE, and also passing through the centre of the needle. As

shown in the diagram the coil C was placed with its axis parallel to AF and its centre on the line DE. C' had its axis in the line AF, and the relative distances of the coils from the magnetometer needle were so adjusted that the magnetic effect of the current passing through the coils was zero at the needle, although the current flowing was made many times greater than that used in the experiments.

The magnet for which the induction correction was to be determined was then placed in one of the coils and the deflection read while as yet no current flowed. A field of about $\frac{1}{10}$ of a C.G.S. unit was then produced by passing a current, and the deflection was once more read. The current was then reversed, and the deflection again noted. The same operations were then repeated with greater and greater currents until a field of from 1 to 2 units had been reached. The magnet was then transferred to the other coil, and a similar series of observations made. It was found that a field of considerably greater intensity than the highest thus used is required to produce any permanent change of the magnetic moment of hard-tempered magnets. The increase of magnetic moment being plotted as ordinate of a curve, the corresponding abscissa of which was the field intensity, enabled the change produced by the earth's field to be obtained by interpolation as described above (p. 18).

A comparison of the results obtained with the two coils showed that the percentage change of deflection produced by the field was smaller for the coil C than for the coil C'. This was undoubtedly due to change of magnetic distribution, the effect of which on the deflection is opposite in the two cases. Assuming that the magnet has an effective half-length λ, the deflection in the first case is

given by (1) and in the other by (2). Thus by using the coils in the two positions as described, the change of distribution as well as the change of moment can be approximately estimated. The plan of two coils has also the advantage of allowing the change of magnetic moment to be obtained free from any error caused by want of exact compensation between the two coils of their direct effect upon the needle.

The results of the experiment showed that to make the effect of induction small the magnet should be hard tempered, and its length should be at least 40 times its diameter. The results are shown in the table on p. 35 below.

The effects of variations in the intensity and direction of the earth's magnetic field were quite marked. The latter showed itself by changes of the magnetometer zero, which were eliminated by reading the zero before and after each deflection, and by reversing the magnets. The effect of change of intensity was allowed for by observing the period of a permanent magnet kept suspended for the purpose. This period was observed at the beginning of the experiment, after the deflection experiment, and again after the oscillation experiment. The necessary correction was estimated from the results and applied. It will be observed that the effect of diurnal variation is quite perceptible. The results in the table on p. 34 are tabulated in the order in which they were obtained, and it will be noticed that the earlier results of each day are generally the smaller. On some occasions on account of magnetic storms it was found impossible to obtain results at all. This was notably the case on Sept. 1, 1885.

The results of this determination are shown in the fol-

lowing two tables. The variation of the effect of induction on the magnetic moment with different ratios of the length of the deflecting magnet to its diameter is shown in the curve of Fig. 6.

Curve illustrating the effect of Ratio of Length to Diameter on the Inductive Coefficient.

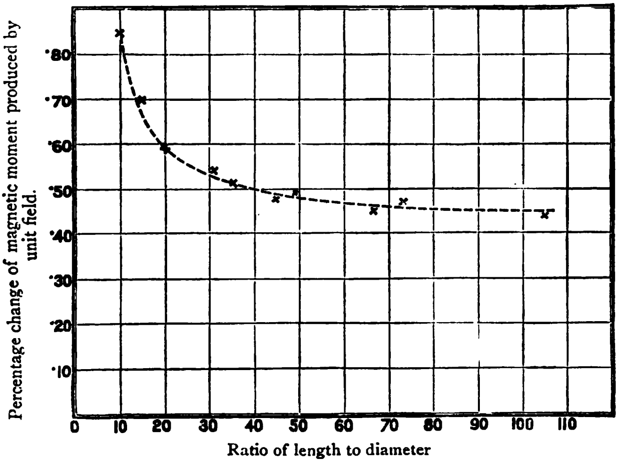

FIG. 6.

It will be observed that the effect of induction diminishes, rapidly at first, then more and more slowly, towards a constant value of about ·4 per cent. for unit field for glass-hard magnets of the kind of steel experimented on.

TABLE I.

Date, 1885.	Number of deflector.	Length of deflector, in centimetres.	Diameter of deflector, in centimetres.	Weight of deflector, in grammes.	Distance of centre of deflector from the magnetometer-needle, in cms. (East and west positions.)	Distance of centre of deflector from the magnetometer-needle, in cms. (North and south positions.)	Distance of the scale from the magneto-meter mirror.	Effective length of the deflector.	Magnetic moment per gramme of the deflector.	Horizontal intensity in C.G.S. units.	Mean of each set of results.	Remarks.
May 27......	1	8·03	0·25	3·054	32·06	28·75	108·7	6·91	44·9	·1520	......	
,, 27......	2	8·05		3·063	,,	,,	,,	7·10	58·5	·1524	......	
,, 28......	3	8·05		3·075	,,	,,	,,	6·31	54·1	·1521	......	
,, 29......	4	8·05		3·067	,,	,,	,,	7·11	52·3	·1522	·1522	
June 5.........	5	4·00	0·25	1·526	30·00	25·125	,,	3·12	35·2	·1524	......	
,, 5.........	6	3·00		1·522	,,	,,	,,	3·72 (?)	33·7	·1524	......	
,, 5.........	7	4·00		1·525	,,	,,	,,	2·33	31·7	·1527	·1524	
June 10	8	14·933	0·25	5·646	51·90	38·85	,,	13·22	55·8	·1527	......	
,, 11	9	15·030		5·727	,,	,,	,,	12·82	64·3	·1525	......	
,, 11	10	15·021		5·666	,,	,,	,,	13·58	54·5	·1527	·1526	
Aug. 21......	11	10·01	0·2	2·318	35·00	30·00	128·9	9·14	71·0	·1526	......	Double deflector method.
	12	10·01		2·336					62·7			
,, 26......	11	10·01		2·318	35·00	30·00	128·9	8·98	71·0	·1527	......	
	12	10·01		2·336					62·7			
,, 31......	13	10·000		2·318	35·00	30·00	128·9	9·14	70·0	·1526	·1526*	
	14	10·005		2·336					61·8			

* Corrected to noon for diurnal variation.

TABLE II.—Showing the effect of Length and of Hardness on the Induction-Coefficient of Magnets.

Length of bar in centimetres.	Ratio of length to diameter.	Unit field. Apparent percentage increase of moment for unit field: side-on position.	Unit field. Apparent percentage increase of moment for unit field: end-on position.	Mean of numbers in columns 3 and 4.	Magnetic moment per gramme.	Remarks.
3	10	0·80	0·90	0·85	27	Glass hard.
4	16	0·67	0·73	0·70	32	,,
4	16	0·67	0·70	0·69	35	,,
6	20	0·51	0·67	0·59	36	,,
7	31	0·51	0·58	0·54	39	,,
8	32	0·51	0·58	0·54	54	,,
8	32	0·51	0·58	0·54	52	,,
10	34	9·46	0·56	0·51	40	,,
10	44	0·40	0·56	0·48	43	,,
7	47	0·46	0·51	0·49	57	,,
10	50	0·44	0·58	0·51	67	,,
10	50	0·48	0·54	0·51	60	,,
10	50	0·46	0·55	0·51	53	,,
10	50	0·46	0·52	0·49	71	,,
10	50	0·46	0·56	0·51	60	,,
10	67	0·41	0·51	0·46	65	,,
7	73	0·41	0·50	0·47	64	,,
10	105	0·42	0·45	0·43	66	,,
10	34	0·47	0·53	0·50	41·5	Glass hard.
10	34	0·63	0·67	0·65	44·5	Yellow.
10	34	0·84	0·98	0·91	54·1	Blue.
10	48	0·32	0·40	0·36	45	Glass hard.
10	48	0·43	0·55	0·49	46	Yellow.
10	48	0·53	0·67	0·60	71	Blue.

The method given above (p. 28) for the determination of the correction for the non-uniform magnetization of the deflecting magnet, gives of course only a first approximation to the true correction, but under the condition that the length of the bar is sufficiently small in comparison with the distance r, say from $\frac{1}{5}$ to $\frac{1}{10}$ of r, and on the supposition that the magnet is reversed at the position on either side of the needle (Fig. 1), it is generally sufficient.

In general no such point as a "centre of 'gravity' of magnetic polarity," that is, a point at which the whole of the free magnetism in the actual distribution in each half of the magnet may be supposed to be concentrated, so as to have the same action on the needle as the actual distribution has, in strictness exists; but, in dealing with the equilibrium or motion of a magnet in a uniform field, and therefore acted on by a magnetic couple, or system of couples, we may regard the centre of inertia of a material system following the same law of distribution as that of the magnetic matter of either half of the magnet as the "centre of mass" of that magnetic matter. This point or "pole" is the centre of the horizontal parallel magnetic forces acting on the particles of ideal free magnetic matter of either half of the magnet when it is hung horizontally in the earth's field, as in the vibration experiments, and is deflected from the magnetic meridian, or of those acting on each magnetism of the needle in the deflection experiments. This is the most proper use of the term "pole," and not that in which it is regarded as a "centre of 'gravity' of magnetic polarity" in the sense just explained.

A method of observation and reduction which amounts to determining the distribution of ideal magnetic matter over the surface of the magnet which has the same external magnetic action as the actual distribution, is given in

Maxwell's *Electricity and Magnetism*, vol. ii., chap. vii., where also will be found full information as to corrections necessary in an exact determination of H.

Let two deflections be taken (Fig. 1) by reversing the deflecting magnet at a distance r_1 on the west side of the needle, and similarly two deflections at the same distance on the east side, and let D_1 be the mean of the tangents of these four deflections. Let this process be repeated at a second distance r_2, and let D_2 be the mean tangent for that distance. It is easy to prove that, approximately :

$$\frac{2M}{H} = \frac{r_1^5 D_1 - r_2^5 D_2}{r_1^2 - r_2^2} \quad . \quad . \quad . \quad . \quad (13)$$

It has been shown (Maxwell's *Electricity and Magnetism*, vol. ii., p. 101 ; or Airy's *Magnetism*, p. 68) that if approximately $r_1 = 1{\cdot}32\ r_2$, the effect of errors in the observed deflections on the value of M/H will be a minimum for these distances.

If long thin bars are used in the determination of H, their magnetic distribution could first be determined very accurately by Rowland's method (Chap. X. below), and the proper corrections applied. On the other hand, short thick bars of hard steel have the advantage of giving greater magnetic moment for a given length, and they can therefore be placed at a greater comparative distance from the needle, so that the correction for the distribution becomes of less importance. So far, then, as the deflection experiments are concerned it is better to use thick short magnets of the hardest steel, and to place them at such a distance from the needle that the error caused by neglecting λ becomes vanishingly small. On the other hand, the magnets must be sufficiently long and thin to render it possible to determine with accuracy their moments of

inertia, and therefore to reduce correctly the results of the vibration experiments.

When λ is small in comparison with r, we may use the approximate formula

$$M = \frac{r^3}{2} H \tan \theta \quad . \quad . \quad . \quad . \quad (14)$$

for the position shown in Fig. 1; or

$$M = r^3 H \tan \theta \quad . \quad . \quad . \quad . \quad (15)$$

for the position shown in Fig. 2.

A magnetic survey of horizontal force in the neighbourhood of a place for which H has been determined may very readily be made with one of the magnets used in the deflection experiments, by simply observing its period of vibration at the various places for which a knowledge of H is desired. The magnetic moment M of the magnet being of course known from the previous experiments, H can be found by equation (5) or (7) above.

By keeping a magnetometer set up with lamp and scale in readiness, the magnetic moments of large magnets can be found with considerable accuracy by placing them in a marked position, at a considerable distance from the needle, and observing the deflection produced. By having a graduated series of distances, for each of which the constant $\frac{1}{2} r^3 H$ or $r^3 H$, as the case may be, by which $\tan \theta$ must be multiplied to give M, has been calculated, the magnetic moments can be very quickly read off.

The magnetic moments of large magnets of hard steel well magnetized can be determined very conveniently with considerable accuracy by hanging them horizontally in the earth's field, and determining the period of a small oscillation about the equilibrium position. They should be hung

by a bundle of as few fibres of unspun silk as possible, at least six feet long, so that the effect of torsion may be neglected. The suspension thread should carry a small cradle or double loop of copper wire, on which the magnet may be laid to give it stability, and to allow of its being readily placed in position or removed. Two vertical marks are fixed in the meridian plane containing the suspension thread, and the observer placing his eye in their plane can easily tell very exactly when the magnet is passing through the equilibrium position, and so determine the period. Or a north and south line may be drawn on the floor or table under the magnet, and the instant at which the magnet is parallel to this line observed by the experimenter by standing opposite one end of the magnet and looking from above. The value of M is given in terms of H by equation (3) above.

Care must of course be taken to avoid undue disturbance from currents of air, and to prevent the magnet, when being deflected from the meridian, from acquiring any pendulum swing under the action of gravity. The deflection from the meridian should be made with another magnet brought, with its length along the east and west line through the centre of the suspended magnet, near enough to produce the requisite deflection, and then withdrawn in the same manner.

The value of H can be readily found with fair accuracy by the ballistic method described below, pp. 317—321. It can also be obtained, of course, by measuring by electrolysis a current flowing through a properly adjusted standard galvanometer, the centre of which is at the place for which H is required. See Chap. IV. below for the elementary theory of a standard galvanometer, and the mode of setting up and using the instrument, and Chap. VII. for the measurement of currents by electrolysis.

CHAPTER III.

ABSOLUTE UNITS OF MAGNETIC POLE, MAGNETIC FIELD, AND ELECTRIC CURRENT.

IN the preceding investigation nothing has been said as to the units in which the quantities M and H are measured. It will be convenient, before proceeding further, to state how the absolute unit of current is defined in the absolute *electromagnetic* system now generally adopted for most electrical measurements. This definition involves those of the absolute units of magnetic pole and magnetic field, which therefore must be considered first.

In the *electromagnetic system* of measurement, all magnetic and electrical quantities are expressed in units which are derived from a magnetic pole chosen as the pole of unit strength. This unit pole might be defined in many ways: but in order to avoid the fluctuations to which most arbitrary standards would be subject, and to give a convenient system in which work done in the displacements of magnets or conductors, relatively to magnets or to conductors carrying currents, may be estimated without the introduction of arbitrary and inconvenient numerical factors, it is connected by definition with the absolute unit of force. It is defined as *a pole which, if*

*placed at unit distance from an equal and similar pole would be repelled with unit force.** The poles referred to in this definition are purely ideal, for we cannot separate one pole of a magnet from the opposite pole of the same magnet: but we can by proper arrangements obtain an approximate realization of the definition. Suppose we have two long, very thin, straight, steel bars, which are uniformly and longitudinally magnetized; their poles may be taken as at their extremities; in fact, the distribution of magnetism in them is such that the magnetic effect of either bar, at all points external to its own substance, would be perfectly represented by a certain quantity of one kind of imaginary magnetic matter placed at one extremity of the bar, and an equal quantity of the opposite kind of matter placed at the other extremity. We may imagine, then, these two bars placed with their lengths in one line, and their blue poles turned towards one another, and at unit distance apart. If their lengths be very great compared with this unit distance, say 100 or 1000 times as great, their red poles will have no effect on the blue poles comparable with the repulsive action of these on one another. But there will be an inductive action between the two magnets which will tend to diminish the mutual repulsive force of the two blue poles, and this we cannot in practice get rid of. The magnitude of this inductive effect is, however, less for hard steel than for soft steel, and we may therefore imagine the steel of the magnets so hard that the action of one on the other does not appreciably affect the distribution of magnetism in either. If, then, two equal blue poles repel one another with unit force, each according to the definition has unit strength.

* The medium between the poles is supposed to be air.

The magnitude of unit pole is by the above definition made to depend on unit force. Now unit force is defined, according to the system of measurement of forces founded on Newton's Second Law of Motion (the most convenient system), as that force which, acting for unit of time on unit of mass, will give to that mass unit of velocity. The unit pole is thus based on the three fundamental units of length, mass, and time. According to the recommendations of the B.A. Committee, and the resolutions of the Paris Congress, it has been resolved to adopt generally the three units already in very extended use for the expression of dynamical, electrical, and magnetic quantities, namely, the centimetre as unit of length, the gramme as unit of mass, and the second as unit of time; and these units are designated by the letters C.G.S. With these units, therefore, unit force is that force which, acting for one second on a gramme of matter, generates a velocity of one centimetre per second. This unit of force has been called a *dyne*. The unit magnetic pole, therefore, in the C.G.S. system of units is that pole which, placed at a distance of one centimetre from an equal and similar pole, is repelled with a force of one dyne. Each of the poles of the long thin magnets of our example above is therefore a pole of strength equal to one C.G.S. unit, if the mutual force between the poles is one dyne.

The magnetic moment M of any one of the deflecting magnets is equal to the strength of either pole multiplied into the distance between the poles, which for magnets of such great length in comparison with their thickness is nearly enough the actual length of the magnet. Therefore either pole has a strength of $M/2\lambda$ units. If r and λ are measured in centimetres, and W in grammes, the strengths

of the magnetic poles deduced from equation (4) or (6) will be in C.G.S. units.

A magnetic field is the space surrounding a magnet or a system of magnets, or a system of conductors carrying currents, at any point of which, if a magnetic pole were placed, it would be acted on by magnetic force. The intensity of the magnetic field due to any system of magnets or of conductors carrying currents, or any combination of such systems, is measured at every point by the force which a unit magnetic pole would there experience, and the direction and magnitude of this force can, theoretically, be calculated if the magnetic distribution is given. Hence from the definition of unit magnetic pole we get at once a definition of magnetic field of unit intensity. *A magnetic field of unit intensity*, or, as we may call it, *unit magnetic field, is that field in which unit magnetic pole is acted on by unit force*, and in the C.G.S. system, therefore, it is that field in which unit magnetic pole is acted on by a force of one dyne. In the theory of the determination of H, given above, the horizontal force on either pole of the magnetometer needle due to the horizontal component of the earth's field is taken as $M'H/2\lambda'$, and again the horizontal force on either pole of the deflecting magnet as $MH/2\lambda$. H therefore measures in units of magnetic field intensity the horizontal component of the earth's field. By formula (5) or (7), when r and λ are taken in centimetres, and W in grammes, H is given in dynes; that is, it is the number of dynes with which a unit red pole would be pulled horizontally towards the north, and a unit blue pole towards the south if acted on only by the earth's magnetic field. In the magnetic field due to a single magnetic pole, the direction of the resultant magnetic force at any point is in the straight line

joining the point with the pole producing the field, and the magnitude of this force, which measures the intensity of the field at the point, is the force which a unit pole would there experience, and is measured by the strength of the pole producing the field, divided by the square of its distance from the point in question.

According to the theory of electromagnetic action given by Ampère,* a plane closed circuit carrying a current in a magnetic field, and of very small dimensions in comparison with its distance from any part of the magnetic system producing the field, is equivalent, as regards magnetic action, to a small magnet, having a magnetic moment directly proportional to the strength of the current and to the area of the circuit, placed with its length at right angles to the plane of the circuit at some point within it, and so turned that to an observer towards whom the red pole is pointing, the current circulates in the direction opposite to the motion of the hands of a watch. If we measure the current in such units that the magnetic moment of the equivalent magnet is equal numerically to the strength of the current multiplied by the area of the circuit, we define unit current (1) as *that current which flowing in a small plane circuit, is equivalent,*† *in magnetic action to a small magnet of moment numerically equal to the area of the circuit.*

From this result for a small plane closed circuit it follows that the mutual action between a finite closed circuit of any form carrying a current in a magnetic field and the magnetic system producing the field is the same

* See for a fuller treatment of this subject the author's treatise on *The Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

† A circuit and magnet equivalent in one medium are not necessarily equivalent in another medium. It is assumed here as elsewhere that the medium is air.

as that which would exist if the circuit were replaced by any magnetic shell (that is a thin material surface magnetized so as to have, on one side, blue magnetism, and on the other side red magnetism), whose bounding edge occupies the position of the circuit, provided the shell does not between any two positions include any part of the magnetic system, and is magnetized in the proper direction, and so that the magnetic moment of every small part is equal to the strength of the current multiplied into the area of that part.

To see this it is only necessary to conceive the circuit as the bounding edge of a net formed by two sets of crossing conductors and to suppose round every mesh an independent current to circulate of the same strength and in the same direction as the current in the circuit. Each mesh may be taken so small that it may be regarded as plane, and it is therefore equivalent in magnetic action to a small magnet specified as above. By taking the meshes infinitely small we get by the equivalent small magnets the magnetic shell just described. But it is plain that in every conductor which is common to two meshes, there will be two equal and opposite currents, which cancel one another, and that this will hold for all parts of the network except those included in the bounding edge. Hence the shell may have any geometrical disposition provided its bounding edge coincide with the circuit, and it do not include between two possible positions any part of the magnetic system the mutual action between which and the shell is considered.

By considering the work (p. 72) done in any small displacement of an element of the boundary of an equivalent shell, it can be shown that every element of the circuit is acted on directly by a force, at right angles to the

plane through the element and the direction of the resultant magnetic force at its centre, the magnitude of which is obtained by multiplying together the length of the projection of the element on a plane at right angles to the direction of the resultant magnetic force, the magnitude of this force, and the strength of the current. Thus if (Fig. 7) ds denote the length of the element, I the resultant magnetic force, or, which is the same, the intensity of the field at the element, θ the angle between the length of the element and the direction $A\,B$ of the magnetic force, C the

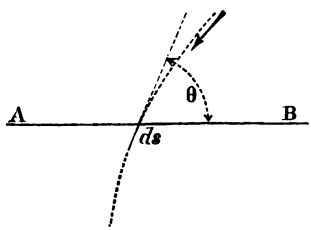

FIG. 7.

strength of the current, and $d\,F$ the force on the element, we have $dF = C\,I\,ds \sin\theta$. If we take $\theta = 90^\circ$, and $I = 1$, we get $dF = C\,ds$, and hence the definition :—(2) *Unit current is the current flowing in an element of a conductor which placed at right angles to the direction of the resultant force in a field of unit intensity at the element, is acted on by an electromagnetic force, the amount of which per unit of length of the element is equal to unity.* If an observer be supposed immersed in the current so that it flows from his feet to his head, and to look along the line of the resultant magnetic force in

the direction in which it would move a blue magnetic pole, the element if free to move would do so towards his right hand.

If we suppose the magnetic system to consist of a single magnetic pole, and put r for the distance of the pole from the element, and μ for the strength of the pole, I becomes μ/r^2, and we have $dF = C\mu \,.\, ds \,.\, \sin\theta/r^2$.

Considering now the opposite aspect of the stress in this case, the action of each element of the circuit on the pole is, by Newton's law, that action and reaction are equal and opposite, a force equal to dF reversed, and therefore acting *in the same line*. But this is equivalent to an equal and similar parallel force acting in a line through the pole; together with a couple, the moment of which is $r\,dF$, acting in the plane of r and dF. For a closed circuit the resultant moment of the couples for all the elements is zero, and the total action is the same as if the forces alone acted on the pole.

From these considerations we may define unit current as (3) *that current which, flowing in a thin wire forming a circle of unit radius, acts on a unit magnetic pole placed at the centre with unit force per unit of length of the circumference*, that is, with a total force of 2π units; or which produces at the centre of the circle a magnetic field of 2π units of intensity. Thus in the C.G.S. system unit current is that current which flowing in the circle of unit radius acts on a unit magnetic pole at the centre with a force of 2π dynes.

This force acts along the axis of symmetry towards one side or the other of the plane of the circle, according to the nature of the pole and the direction of the current, and its direction may be found for any case by remembering that the earth may be imagined to be a magnet turned

into position by the action of a current flowing round the magnetic equator in the direction of the sun's apparent motion.

These three definitions of unit current are equivalent, and in the applications which follow we shall use that which is most convenient in any particular case.

CHAPTER IV.

MEASUREMENT OF A CURRENT IN ABSOLUTE UNITS AND PRACTICAL CONSTRUCTION OF A STANDARD GALVANOMETER.

IF we take next the simple case of a single wire bent round into a circle and fixed in the magnetic meridian, with a magnet, whose dimensions are very small in comparison with the radius of the circle, hung by a torsionless fibre so as to rest horizontally with its centre at the centre of the circle, we may suppose that each pole of the magnet is at the same distance from all the elements of the wire. A current flowing in the wire acts, by Ampère's theory, with a force on one pole of the needle towards one side of the plane of the circle, and on the other pole with an equal force towards the other side of that plane. The needle is thus acted on by a couple tending to turn it round, and it is deflected from its position of equilibrium until this couple is balanced by the return couple due to H. Let us suppose the strength of each pole of the needle to be m units, r the radius of the circle, and C the strength of the current in it. Then by Ampère's law we have for the whole force without regard to sign, exerted on either pole of the needle by the current, the value $2\pi r \,.\, Cm/r^2$ or $2\pi mC/r$. If l be the distance between the poles the couple is $2\pi Cml/r$ before any

deflection has taken place. After the needle has been deflected through the angle θ the arm l of the couple has become $l \cos \theta$, and therefore the couple $2\pi C m l \cos \theta / r$: and the return couple due to H is $m H l \sin \theta$. Hence we have equilibrium when

$$C m \frac{2\pi}{r} l \cos \theta = m H l \sin \theta$$

and therefore

$$C = \frac{H r}{2\pi} \tan \theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

if θ be the observed angle at which the needle rests in equilibrium when deflected as described from the magnetic meridian. If instead of a single circular turn of wire we had N turns occupying an annular space of mean radius r, and of dimensions of cross-section small compared with r, we should have

$$C = \frac{H r}{2\pi N} \tan \theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In practice the turns of wire of the tangent galvanometer may not be all contained within such an annular space. It is necessary then to allow for the dimensions of the space occupied by the wire. For a coil made of wire of small section we may suppose that the actual current flowing across a unit of area is everywhere the same. Hence if C be the current strength in each turn, and n the number of turns in unit area, we have for the current crossing the area A of an element E the value $n\,C\,A$. Taking a section of the coil through its centre, at right angles to its plane, and supposing the annular space of rectangular section, let $B\,C$ (Fig. 8) be a radius drawn from the centre C in the plane cutting the coil

into two equal and similar coils, and taking $C\,D\;(=y)$ and $D\,E\;(=x)$ at right angles to one another, we have $A = dx\,dy$ and $CE^2 = x^2 + y^2$. Hence the force exerted on a unit magnetic pole at the centre C by an element of area A, and of length ds, supposed at right angles to

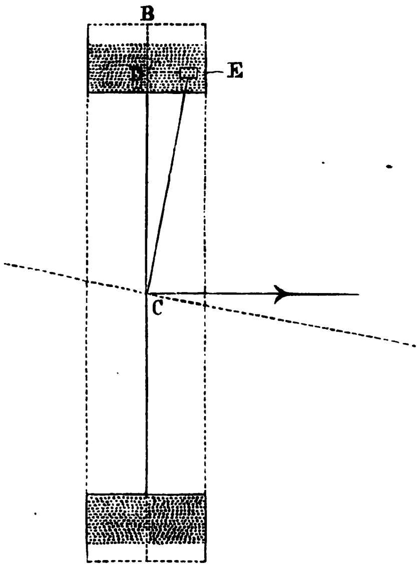

FIG. 8.

the plane of the paper, is $n\,C\,dx\,dy/(x^2 + y^2)$ in the direction at right angles to $C\,E$ and in the plane of the paper. The component of this force at right angles to $B\,C$ is $n\,Cy\,dx\,dy/(x^2 + y^2)^{\frac{3}{2}}$. Hence if we call dF the component in this direction of the whole ring at right

angles to the plane of the paper of which this element is the cross-section we have

$$dF = \frac{2\pi n C y^2 dx dy}{(x^2 + y^2)^{\frac{3}{2}}}.$$

Hence for the whole force at right angles to $B\ C$ we have

$$F = 2\pi n C \int_{-b}^{b} \int_{r-c}^{r+c} \frac{y^2 dx dy}{(x^2 + y^2)^{\frac{3}{2}}},$$

where r is the mean radius of the coil, $2\ b$ its breadth, and $2\ c$ its depth in the plane of the circle.

Integrating, and putting N for the whole number of turns $4\ n\ b\ c$, we get

$$F = \pi N C \frac{1}{c} \log \frac{r + c + \sqrt{(r + c)^2 + b^2}}{r - c + \sqrt{(r - c)^2 + b^2}} \quad . \quad (3)$$

If θ be the angle at which the deflecting couple is equilibrated by the return couple due to H, we have as before the equation

$$F = H \tan \theta.$$

Hence, substituting the above value for F and solving for C, we have finally

$$C = \frac{H \tan \theta}{\pi N \frac{1}{c} \log \frac{r + c + \sqrt{(r + c)^2 + b^2}}{r - c + \sqrt{(r - c)^2 + b^2}}} \quad . \quad (4)$$

When the value of r is great in comparison with b and c this reduces to the equation

$$C = \frac{Hr \tan \theta}{2\pi N} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which we found before by assuming all the turns to be contained in a small annular space of radius r. In

practice, in galvanometers used as standards for absolute measurements, generally neither b nor c is so great as $\frac{1}{10}$ of r, and the needle cannot be made infinitely short; hence in these cases the difference between the values given by equations (4) and (5) is well within the limits of experimental error, and the correction need not be made. The value of C given by (5) is then to be used.

When the dimensions of the coil are measured in centimetres, and H is taken in C.G.S units the value of C is given by (4) or (5) in C.G.S units of current strength. In this investigation the suspension fibre has been supposed torsionless. If a single fibre of unspun silk is used as described below for this purpose, its torsion may for most practical purposes be safely neglected. The error produced by it may however be easily determined and allowed for by turning the needle, supposed initially in the magnetic meridian, once or more times completely round, and noting its deviation from the magnetic meridian in its new position of equilibrium. The amount of this deviation, if any, may be easily observed by means of the attached index and divided circle, or scale and reflected beam of light used as described below, to measure the deflections of the needle. From the result of this experiment the effect of torsion for any deflection may be calculated in the following manner.

Let a be the angular deflection, in radian * measure, of the magnet from the magnetic meridian produced by turning the magnet once round, then the angle through which the thread has been twisted is $2\pi - a$. The couple produced by this torsion has for moment $Hlm \sin a$.

* A *radian* is the angle subtended at the centre of a circle by an arc equal in length to the radius. It has generally been called in books on trigonometry hitherto by the ambiguous name *unit angle in circular measure*.

Hence, by Coulomb's law of the proportionality of the force of torsion to the twist given, we have for the couple corresponding to a deflection θ the value

$$\frac{\theta}{2\pi - \alpha} H m l \sin \alpha.$$

If then under the action of a current in the coil the deflection of the needle is θ, the equation of equilibrium is

$$Cm \frac{2\pi}{r} l \cos \theta = m H l \left(\sin \theta + \frac{\theta}{2\pi - \alpha} \sin \alpha\right)$$

and therefore instead of (5) we have

$$C = \left(1 + \frac{\theta}{2\pi - \alpha} \frac{\sin \alpha}{\sin \theta}\right) \frac{Hr}{2\pi N} \tan \theta \quad . \quad . \quad (6)$$

If α be an angle of say $1°$, and θ be $45°$, $\theta/(2\pi - \alpha)$ is very nearly $1/8$ and $\sin \alpha / \sin \theta$ is $1/(57{\cdot}3 \times {\cdot}707)$ or $1/40{\cdot}5$. Hence

$$C = \left(1 + \frac{1}{324}\right) \frac{Hr}{2\pi N}.$$

The error therefore is somewhat less than $\frac{1}{3}$ per cent.

The determination of H and the measurement of a current in absolute units, can be effected simultaneously by the method devised by Kohlrausch, and described in the *Philosophical Magazine*, vol. xxxix. 1870. This method consists essentially in sending the current to be measured through two coils, of which all the constants are accurately known. One of these is the coil of a standard galvanometer, the other is a coil hung by a bifilar suspension, the wires of which convey the current into the coil. The latter coil rests in equilibrium when no current is passing through it, with its plane in the magnetic meridian. When a current is sent through it,

it is acted on by a couple due to electromagnetic action between the current and the horizontal component of the earth's force, which tends to set it with its plane at right angles to the magnetic meridian; and this couple is resisted by the action of the bifilar. The coil comes to rest, making a certain angle with the magnetic meridian, and, as the couple exerted by the bifilar suspension for any angle is supposed to have been determined by experiment, a relation between the value of H and the value of the current is obtained. But, as the same current is sent through the coil of the standard galvanometer, the observed deflection of the needle of that instrument gives another relation between H and C. From the two equations expressing these relations the values of H and C can be found. Full details of the construction of Kohlrausch's apparatus and of the calculation of its constants will be found in the paper above referred to.

In this method it is assumed that the value of H is the same at both instruments, an assumption which for rooms not specially constructed for magnetic experiments cannot safely be made. An instrument which is not liable to this objection has been suggested by Sir William Thomson. A short account of this instrument and its theory will be found in Maxwell's *Electricity and Magnetism*, vol. ii. p. 328, and in the author's Treatise on the *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

The galvanometer may with advantage in some particulars be composed of two equal coils of which the radii are great in comparison with the transverse dimensions, placed parallel to one another so that their line of centres is normal to their mean planes. A short needle is suspended symmetrically with respect to the coils on the

line of centres, and the current is sent through the coils arranged in series so that the magnetic action of each tends to turn the needle round in the same direction. The wire passing from one coil to the other is placed close to that passing back to the electrodes of the instrument, which are preferably two well-insulated wires twisted together, and so long that no loop of the circuit, caused by binding screws or any other contact-making arrangement, is near the needle. Thus no magnetic action on the needle is produced except that due to the current in the coils.

Let N be the number of turns of wire in each coil, r the mean radius of each, $2a$ their mean distance apart, and C the current producing an angular deflection θ, then

$$C = \frac{(r^2 + a^2)^{\frac{3}{2}} H \tan\theta}{4\pi r^2 N} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Helmholtz has shown that for an instrument of this construction the mathematical theory * (for a needle not infinitely short) is greatly simplified if $r = 2a$, and he has accordingly advised that the instrument should be so constructed. In ordinary practice however a single coil (or, as described below, p. 61, a long coil having only one layer) with needle at the centre gives results which are sufficiently accurate, and it is much more easy to construct.

It is evident from the equation that the greater a is made the smaller is the sensibility. By making the coils movable therefore along a graduated bar to different distances apart, an instrument (though not conveniently a

* See *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

standard instrument) having a sensibility variable at will through a considerable range can be constructed.

The accuracy of the measurements of currents, made according to the method of which the elementary theory is given above, of course altogether depends on the careful adjustment of the standard galvanometer, and the care and skill of the observer.

The standard galvanometer should be of such a form that the values of its indications can be easily calculated from the dimensions and number of turns of wire in the coil. Such a galvanometer can be made by any one who can turn, or can get turned, a wooden or brass (see p. 62) ring with a rectangular groove round its outer edge to receive the wire. It is indeed to be preferred that the experimenter should at least perform the winding of the coil and the adjustments of the needle, &c., himself, to make sure that errors in counting the number of turns or in determining the length of the wire, or in placing the needle at the centre of the coil, are not made. If there are to be several layers of wire, the breadth and depth of this groove ought to be small in comparison with its radius, and each should not be greater than $\frac{1}{10}$ of the mean radius of the coil, which should be at least 15 cms.

The gauge of the wire with which the coil is to be wound must depend of course on the purposes to which the instrument is to be applied, but it should be good well insulated copper wire of high conductivity, and not so thin as to run any risk of being injured by the strongest currents likely to be sent through the instrument. For the exact graduation of current as well as potential galvanometers directly by means of the standard instrument, it is convenient to make it have two coils—one of comparatively high, the other of low resistance. The

latter may conveniently in some cases in which great accuracy is not required, be a simple hoop of say 15 cms. radius, made of copper strip 1 cm. broad and 1 mm. thick. As however the distribution of the current in a massive conductor is uncertain in consequence of want of homogeneity in the material, and it is besides difficult to allow exactly for any irregularity that may exist where the ends are led out, so that it is impossible to determine by calculation the exact constant of such a coil, it is better to use instead several turns of thick wire. Each spire of the coil may then be regarded as explained above, as a circular conductor coinciding with its circular axis.

To form electrodes to which wires can be attached the ends of the strip are brought out side by side in the plane of the ring with a piece of thin vulcanite or paper between for insulator. Insulated wires are soldered to the ends of the circle thus arranged, and are twisted together for a sufficient distance to prevent any direct effect on the needle from being produced by a current flowing in them.

In constructing the fine wire coil the operator should first subject the wire to a moderate stretching force, and then carefully measure its electrical resistance and its length. He should then wind it on a moderately large bobbin, and again measure its resistance. If the second measurement differs materially from the first, the wire is faulty and should be carefully examined. If no evident fault can be found, on the removal of which the discrepance disappears, the wire must be laid aside and another substituted. When the two measurements are found to agree the wire may then be wound on the coil. For this purpose the ring may either be turned slowly round in a lathe or on a spindle, so as to draw off the wire from the

bobbin, also mounted so as to be free to turn round. The wire must be laid on evenly in layers in the groove (which may be done with the greatest uniformity by using a self-feeding lathe), and the winding ended with the completion of a layer. Great care must be taken to count accurately the number of turns laid on. The resistance should now be again tested, and if it agrees nearly with the former measurements the coil may be relied on.

The ring carrying the coil thus made should then be fixed to a convenient stand in such a manner that if necessary it can be easily removed. The stand ought to be fitted with levelling screws, so that the plane of the coil may be made accurately vertical. A shallow horizontal box with a glass cover and a mirror bottom should be carried by the stand at the level of its centre. Within this the needle and attached mirror or index are to be suspended. Or, what is more convenient in many cases, a platform should be arranged below the level of the centre a sufficient distance to allow the magnetometer described in Chap. II. above, to be placed with the centre of its needle at the level of the centre of the coil.

The needle should be a single small magnet about a centimetre long, hung by a single fibre of unspun silk about ten cms. long from the top of a tube fixed to the cover of the shallow box, so that the centre of the needle when the coil is vertical is exactly the centre of the coil. To allow of the exact adjustment of the height of the needle, the fibre should be attached to the lower end of a small screw spindle, made so as to be raised or lowered, without being turned round, by a nut working round it above the cap of the tube.

If the instrument is to be used with scale and pointer

(or, as is desirable in some cases, is to be furnished with scale and pointer as well as mirror) the pointer may be made by drawing out a bit of thin glass tube at the blow-pipe into a thread, so thick as to remain nearly straight under its own weight when suspended by its centre. In order that the zero position of the pointer may not be under the coil, the pointer ought to be fixed horizontally with its length at right angles to the needle, so as to project to an equal distance on both sides of it. To test that this adjustment is accurately made, draw a couple of lines accurately at right angles to one another on a sheet of paper. Then suspend a long thin straight magnet over the paper, and bring one of the lines into accurate parallelism with it. Remove then the magnet and put in its place the little needle and attached index. If the index is parallel to the other line the adjustment has been carefully made. The needle may then be suspended in position and the box within which it hangs closed to prevent disturbance from currents of air.

A circular scale graduated to degrees, with its centre just below the centre of the coil and its plane horizontal, is placed with its zero point on a line drawn on the mirror bottom of the box at right angles to the plane of the coil, so that when the needle and coil are in the magnetic meridian the index may point to zero. The accuracy of the adjustment of the zero point is to be tested by finding whether the same current reversed produces equal deflections on the two sides of zero.

To test whether the centre of this divided circle is accurately under the centre of the needle supposed at the centre of the coil, draw from the point immediately under the centre of the needle two radial lines on the mirror bottom, one on each side of the zero point and 45° from

it, and turn the needle round without giving it any motion of translation. If the index lies along these two radial lines when its point is at the corresponding division on the circle the adjustment is correct.

When taking readings the observer places his eye so as to see the index just cover its image in the mirror bottom of the box, and reads off the number of divisions and fraction of a division, indicated on the scale by the position of the index. Error from parallax is thus avoided.

A mirror with attached magnets may be used, as in the magnetometer, instead of the needle and index. Very conveniently a magnetometer with long fibre may be used placed on a platform fixed within the bobbin. A hole slot and plane arrangement cut in the platform for the adjusted position will enable the magnetometer to be taken away, and replaced at pleasure. The adjustment of scale, &c., is the same as that described in Chap. II. above. When a mirror is employed the coil is in the magnetic meridian when equal deflections of the spot of light on the scale on the two sides of zero are produced by reversing any current. The scales used, as has been already remarked, should, if of paper, always be carefully glued to a wooden piece, instead of being, as they frequently are, fixed with drawing pins. Preferably however they should be ruled on glass by any one of the simple methods now available for copying an accurately engraved standard.

The author has had constructed lately a standard galvanometer which seems to have several advantages. It consists of a cylindrical bobbin of about 50 centimetres in diameter and 25 centimetres in length, wound with a single layer of fine wire. The needle (1 cm. long) is suspended at the centre of the bobbin, and the magnetic field produced by a current flowing in the wire is in this

arrangement practically invariable over a distance in any direction at the centre considerably exceeding the length of the needle. Very accurate placing of the needle is thus not necessary, as a displacement of so much as half its length from the central position (an error of adjustment which is practically impossible with the slightest care) produces a quite imperceptible effect in the deflection with any given current.

The distribution of the wire, since there is only one layer, is known with perfect certainty, and hence the constant of the instrument can be calculated with great exactness. At each end of the bobbin is wound one of two equal coils of small transverse dimensions in comparison with their radii. These are of thick copper wire arranged so as to form a Helmholtz double-coil galvanometer of the kind described above (p. 56), available for strong currents.

When the instrument was being designed it was thought desirable to have the bobbin made of some material which could not contain magnetic substances in sufficient quantity to affect the accuracy of measurements of currents flowing in the wire. The fear then felt by the author that the bobbins of brass ordinarily employed for standard galvanometers might very probably contain iron in sufficient quantity to cause disturbance through its inductive magnetization, has since been found by Prof. Thomas Gray to be in part at least justified. The measurements of currents made by a new standard galvanometer were found by him to be so much disturbed by the effect of magnetic substances contained in a brass box surrounding the needle as to be practically useless.

It was resolved therefore to construct a bobbin of wood in such a manner as to avoid all risk of alteration of

figure by warping, or of dimensions through variation in the amount of moisture contained in the wood. Accordingly a sufficient number of pieces of mahogany were cut from a dry well-seasoned board about ½ inch thick. Each piece was about 4 cms. broad, 20 cms. in length, and was cut so as to form a segment of a ring the outside diameter of which was about 50 cms. and the inner diameter about 8 cms. less. Four of these cut so that the grain of the wood ran in different directions in adjoining pieces and placed end to end, gave a complete circular ring, or rather cylinder, ½ inch in length. Above that was placed a similar ring with the grain of the wood in the pieces crossing that in the pieces below, and the pieces themselves overlapping the end joints in the preceding ring. Above that was placed another ring, and so on until the whole bobbin, rather more than 25 cms. in length, had been built up. The whole was then put together under considerable pressure, with glue between the joints. The cylinder thus roughly formed was then turned carefully down to a true cylindrical figure of the size desired, and the pores all over the surface, inside and outside, filled with spirit varnish to prevent the absorption of moisture.

Two edges of wood, projecting slightly beyond the outside cylindrical surface, were fixed on the ends to keep the wire in its place. The coil was then carefully wound, the turns counted, and the wire covered with American cloth to preserve it from injury. The two ends of the thin wire coil were brought out together at one end of the coil to be connected to two electrodes closely twisted together, and several yards in length, by which the instrument could be joined to any circuit in which it might be required. The end of the wire which had to be

carried from the further extremity of the coil was (supposing the coil set up in position) brought along horizontally in a vertical plane through the axis of the coil until it met the other extremity at the termination of the last spire of the coil. The current in this part of the wire of course just compensates by its effect on the needle the effect of the component of current in each element of the spires in the direction of the axis.

The constant of the instrument was calculated by putting in equation (3) or (4) above $c = 0$. Thus if N be the total number of turns in the single layer of the coil, we have

$$C = \frac{(r^2 + b^2)^{\frac{3}{2}} H \tan \theta}{2 \pi N} \quad . \quad . \quad . \quad . \quad (8)$$

For the thick wire coils the equation connecting the current with the deflection is given above (7), p. 56.

We have here described the tangent galvanometer as it is the instrument most easily constructed; but what are called *sine galvanometers* are sometimes employed. In these the coil can be turned round a vertical axis to allow its medial plane to be placed parallel to the needle in the deflected position of the latter. Then if θ be the angle of deflection of the needle (which can be measured by observing the angle through which the coil has been turned from the meridian) we have for the equations of equilibrium (1), (2), &c. above, with $\tan \theta$ replaced by $\sin \theta$. Particulars of the construction of such galvanometers will be found in the author's *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

CHAPTER V.

DEFINITION OF ABSOLUTE UNITS OF DIFFERENCE OF POTENTIAL AND RESISTANCE, AND DERIVATION OF PRACTICAL UNITS—VOLT, OHM, AMPERE, COULOMB. RATE OF WORKING IN AN ELECTRIC CIRCUIT.

TWO conductors of the same material, are at different potentials if on their being put in conducting contact electricity tends to pass from one to the other. A difference of potential may also exist between two conductors of different materials in contact, or may be maintained between two parts of the same conductor by electrical forces. In all cases a difference of potential is measured by the work (p. 73) which would be done by electric forces on unit quantity of positive electricity if, while the potentials remained the same, it were carried from the place of higher to the place of lower potential. Two bodies at different potentials attract one another; and therefore, if one be connected with an electrically insulated plate carried at one end of a delicate balance, and the other with a second insulated plate fixed facing the first at a very short distance from it, these two plates will, if they have been previously put for an instant in conducting contact and balance obtained, attract one another, and the force of attraction may be weighed by

restoring balance. With certain arrangements necessary to ensure accuracy, a balance may be constructed by means of which the difference of potential between two conductors can be measured. Such an instrument has been made by Sir William Thomson, and called by him an Absolute Electrometer.

It is found experimentally by measuring with a delicate electrometer that, between any two cross-sections A and B of a homogeneous wire, in which a uniform current of electricity is kept flowing by any means, there exists a difference of potential, and that if the wire be of uniform section throughout, the difference of potential is in direct proportion to the length of wire between the cross-sections. It is found further, that if the difference of potential between A and B is kept constant, and the length of wire between them is altered, the strength of the current varies inversely as the length of the wire. The strength of the current is thus diminished when the length of the wire is increased, and hence the wire is said to oppose *resistance* to the current; and the resistance between any two cross-sections is proportional to the length of wire connecting them. If the length of wire and the difference of potential between A and B be kept the same, while the cross-sectional area of the wire is increased or diminished, the current is increased or diminished in the same ratio; and therefore the resistance of a wire is said to be inversely as its cross-sectional area. Again, if for any particular wire, measurements of the current strength in it be made for various measured differences of potential between its two ends, the current strengths are found to depend only on, and to be in simple proportion to, the differences of potential so long as there is no sensible heating of the wire. This last statement is Ohm's law. The strength

C of the current flowing in a wire of resistance R, between the two ends of which a difference of potential V is maintained, is given by the equation

$$C = \frac{V}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation the units in which any one of the three quantities is expressed depend on those chosen for the other two. We have defined unit current, and have seen how to measure currents in absolute units; and we have now to show how the absolute units of V and R are to be defined, and from them and the absolute unit of current to derive the practical units—volt, ampere, coulomb, and ohm.

We shall define the absolute units of potential and resistance by a reference to the action of a very simple but ideal magneto-electric machine, of which, however, the modern dynamo is merely a practical realization. First of all let us imagine a uniform magnetic field of unit intensity. The lines of force in that field are everywhere parallel to one another: to fix the ideas let them be vertical. Now imagine two straight horizontal metallic rails running parallel to one another, and connected together by a sliding bar, which can be carried along with its two ends in contact with them. Also let the rails be connected by means of a wire so that a complete conducting circuit is formed. Suppose the rails, slider, and wire to be all made of the same material, and the length and cross-sectional area of the wire to be such that its resistance is very great in comparison with that of the rest of the circuit, so that, when the slider is moved with any given velocity, the resistance in the circuit remains practically constant. When the slider is moved along the rails it cuts across the lines of force, and so long as it moves

with uniform velocity a constant difference of potential will be maintained between its two ends by induction, and a uniform current will flow in the wire from the rail which is at the higher potential to that which is at the lower. If the direction of the lines of force be the same as the direction of the vertical component of the earth's magnetic force in the northern hemisphere, so that a red or north-tending pole placed in the field would be moved downwards, and if the rails run south and north, the current when the slider is moved northwards will flow from the east rail to the west through the slider, and from the west rail to the east through the wire. If the velocity of the slider be increased the difference of potential between the rails, or, as it is otherwise called, the electromotive force producing the current, will be increased in the same ratio ; and therefore by Ohm's law so also will the current.

Generally for a slider arranged as we have imagined, and made to move across the lines of force of a magnetic field, the difference of potential produced would be directly as the field intensity, as the length of the slider, and as the velocity with which the slider cuts across the lines of force. The difference of potential produced therefore varies as the product of these three quantities ; and when each of these is unity, the difference of potential is taken as unity also. We may write therefore $V = I\,L\,v$, where I is the field intensity, L the length of the slider, and v its velocity. Hence if the intensity of the field we have imagined be 1 C.G.S. unit, the distance between the rails 1 cm., and the velocity of the slider 1 cm. per second, the difference of potential produced will be 1 C.G.S. unit.

This difference of potential is so small as to be incon-

venient for use as a practical unit, and instead of it the difference of potential which would be produced if, everything else remaining the same, the slider had a velocity of 100,000,000 cms. per second, is taken as the practical unit of electromotive force, and is called one *volt*. It is a little less than the difference of potential which exists between the two insulated poles of a Daniell's cell.

We have imagined the rails to be connected by a wire of very great resistance in comparison with that of the rest of the circuit, and have supposed the length of this wire to have remained constant. But from what we have seen above, the effect of increasing the length of the wire, the speed of the slider remaining the same, would be to diminish the current in the ratio in which the resistance is increased, and a correspondingly greater speed of the slider would be necessary to maintain the current at the same strength. We may therefore take the speed of the slider as measuring the resistance of the wire. Now suppose that when the slider 1 cm. long was moving at the rate of 1 cm. per second, the current in the wire was 1 C.G.S. unit; the resistance of the wire was then 1 C.G.S. unit of resistance. Unit resistance therefore corresponds to a velocity of 1 cm. per second. This resistance, however, is too small to be practically useful, and a resistance 1,000,000,000 times as great, that is the resistance of a wire, to maintain 1 C.G.S. unit of current in which it would be necessary that the slider should move with a velocity of 1,000,000,000 cms. (approximately the length of a quadrant of the earth from the equator to either pole) per second, is taken as the practical unit of resistance, and called one *ohm*.

In reducing the numerical expressions of physical quantities from a system involving one set of fundamental units

to a system involving another set, as for instance from the British foot-grain-second system, formerly in use for the expression of magnetic quantities, to the C.G.S. system, it is necessary to determine, according to the theory of "dimensions" (first given by Fourier, and extended to electrical and magnetic quantities by Maxwell), for each a certain reducing factor, by substituting in the dimensional formula which states the relation of the fundamental units to one another in the expression of the quantity, the value of the units we are reducing from in terms of those we are reducing to. For example, in reducing a velocity, say from miles per hour to centimetres per second, we have to multiply the number expressing the velocity in the former units by the number of centimetres in a mile, and divide the product by the number of seconds in an hour; that is, we have to multiply by the ratio of the number of centimetres in a mile to the number of seconds in an hour. The multiplier therefore, or *change-ratio* as it has been called by Professor James Thomson, is for velocity simply the number of the new units of velocity equivalent to one of the old units, and may be expressed by the formula L/T, where L is the numerical expression of the old unit of length in terms of the new contained in one of the old, and T is the corresponding expression for the old unit of time. In the same way the change ratio for rate of change of velocity or acceleration is L/T^2; and the change-ratio of any other physical quantity may be found by determining from its definition the manner in which its unit involves the fundamental units of mass, length, and time.

Now the theory of the dimensions of electrical and magnetic quantities, in the electromagnetic system of units, shows (Ch. XI.) that the dimensional formula

for resistance is the same as that for velocity; that in fact a resistance in electromagnetic measure is expressible as a velocity; and hence we may with propriety speak of a resistance of one ohm as 10^9 centimetres per second.

The first experiments for the realization of the ohm were made by the British Association Committee, and later determinations have been made with varying results by experimenters in different parts of the world. The method used by the committee was one suggested by Sir W. Thomson, in which the current induced in a coil of wire revolving round a vertical diameter in a magnetic field was measured by the deflection of a magnetic needle hung at the centre of the coil. The principle of the method is exactly the same as that of the ideal method, with slider and bars, sketched above. According to the results of the committee the ohm is represented approximately by the resistance of a column of pure mercury 104·8 centimetres long, one square millimetre in section, at the temperature 0° C. Coils of an alloy of two parts of silver to one of platinum, which had a resistance of one ohm at a certain temperature, were issued by the committee as standards to experimenters.

Concordant experiments by Prof. Rowland in America, by Lord Rayleigh and by Mr. Glazebrook at Cambridge, and by Messrs. Mascart, de Nerville, and Benoit in Paris, have shown that the B.A. unit is probably about 1·3 per cent. too small. A very careful determination by Lord Rayleigh and Mrs. H. Sidgwick gives ·98677 earth-quadrant per second as its value.* Lord Rayleigh and Mrs. Sidgwick also measured the specific resistance of mercury

* *Phil. Trans. R. S.* 1883.

in terms of the B.A. unit.* Their result, together with their value of the B.A. unit gave 106·21 cms. as the length of a column of pure mercury of section 1 sq. mm. and temp. 0°C. which has a resistance of 1 ohm.

A determination of the ohm in terms of the B.A. unit has recently been made by Messrs. L. Duncan, G. Wilkes, and C. T. Hutchinson at the Johns Hopkins University, Baltimore, with apparatus designed by Prof. H. A. Rowland.† Their result is ·9863 earth-quadrant per second = 1 B.A. unit, and this taken with a redetermination of the specific resistance of mercury made by the last-named two gentlemen,‡ gives 106·34 cms. as the length of the mercury column specified as above.

It has been agreed by the Congress of Electricians held at Paris in 1884, to adopt temporarily as the Legal Ohm, a resistance equal to that of a column of pure mercury 106 cms. in length and one square millimetre in section, at the temperature of 0°C. When a sufficient number of exact determinations of the ohm have been obtained the legal value will no doubt be brought into as close as possible accord with the true value.

It is obvious from equation (1) that if V and R, each initially one unit, be increased in the same ratio, C will remain one unit of current; but that if V be, for example, 10^8 C.G.S. units of potential, or one volt, and R be a resistance of 10^9 cms. per second, or one ohm, C will be one-tenth of one C.G.S. unit of current. A current of this strength—that is, the current flowing in a wire of resistance one ohm, between the two ends of which a difference of potential of one volt is maintained,—has been adopted as the practical unit of current, and called

* *Phil. Trans. R. S.* 1883. See also below, p. 242.
† *Phil. Mag.* Aug. 1889. See also below, p. 242.
‡ *Ibid.* July, 1889.

one *ampere*. Hence it is to be remembered one ampere is one-tenth of one C.G.S. unit of current.

The amount of electricity conveyed in one second by a current of one ampere is called one *coulomb*. This unit, although not quite so frequently required as the others, is very useful, as, for instance, for expressing the quantities of electricity which a secondary cell is capable of yielding in various circumstances. For example, in comparing different cells with one another their capacities, or the total quantities of electricity they are capable of yielding when fully charged, are very conveniently reckoned in coulombs per square centimetre of the area across which the electrolytic action in each takes place.

The magneto-electric machine we have imagined gives us a very simple proof of the relation between the work done in maintaining a current, the strength of the current, and the electromotive force producing it. In dynamics *work* is said to be done *by* a force when its place of application has a component motion *in the direction in which the force acts*, and the work done by it is equal to the product of the force and the distance through which the place of application of the force has moved in that direction. The rate at which work is done by a force at any instant is therefore equal to the product of the force and the component of velocity in the direction of the force at that instant. The work done in overcoming a resistance through a certain distance is equal by this definition to the product of the resistance and the distance through which it is overcome. Among engineers in this country the unit of work generally used, is *one foot-pound*, that is, the work done in lifting a pound vertically against gravity through a distance of one foot, and the unit rate of working is one *horse-power*, that is 33,000 foot-pounds per minute. The

weight of a pound of matter being generally different at different places, this unit of work is a variable one, and is not used in theoretical dynamics. In the absolute C.G.S. system of units, the unit of work is the work done in overcoming a force of one dyne through a distance of one centimetre, and is called one centimetre-dyne or one *erg*. The single word *Activity* has been used by Sir Wm. Thomson as equivalent in meaning to "rate of doing work," or the rate per unit of time at which energy is given out by a working system; and in what follows we shall frequently use the term, or the alternative word *power*, in that sense.

We have seen above (p. 46) that every element of a conductor, carrying a current in a magnetic field, is acted on by a force tending to move it in a direction at right angles to the plane through the element, and the direction of the resultant magnetic force at the element, and have derived from the expression for the magnitude of the force a definition ((2) p. 46) of unit current in the electromagnetic system. From these considerations it follows that a conductor in a uniform magnetic field, and carrying a unit current which flows at right angles to the lines of force, is acted on at every point by a force tending to move it in a direction at right angles to its length, and the magnitude of this force for unit length of conductor, and unit field, is by the definition of unit current equal to unity.

Applying this to our slider in which we may suppose a current of strength C to be kept flowing, say, from a battery in the circuit, let L be the length of the slider, v its velocity, and I the intensity of the field; we have for the force on the moving conductor the value $I L C$. Hence the rate at which work is done by the electromagnetic action between the current and the field is $I L C\, dx/dt$ or $I L C v$, and this must be equal to the rate at

which work would be done in generating by motion of the slider a current of strength C. But as we have seen above, ILv is the electromotive force produced by the motion of the slider. Calling this now E, the symbol usually employed to denote electromotive force, we have EC as the electrical activity, that is, the total rate at which electrical energy is given out in all forms in the circuit.

By Ohm's law this value for the electrical activity may, when the work done is wholly spent in producing heat, be put into either of the two other forms, namely, E^2/R, or C^2R. In the latter of these forms the law was discovered by Joule, who measured the amount of heat generated in wires of different resistances by currents flowing through them. This law holds for every electric circuit whether of dynamo, battery, or thermoelectric arrangement.

We have, in what has gone before, supposed the slider to have no resistance comparable with the whole resistance in the circuit. If it have a resistance r, and R be the remainder of the resistance in circuit, the actual difference of potential between its two ends will not be ILv or E, but $E.R/(R+r)$ (p. 83). The rate per unit of time at which work is given out in the circuit is however still EC, of which the part $EC.r/(R+r)$ is given out in the slider, and the remainder, $EC.R/(R+r)$, in the remainder of the circuit. In short, if V be the actual difference of potential, as measured by an electrometer, between two points in a wire connecting the terminals of a battery or dynamo, and C be the current flowing in the wire, the rate at which energy is given out is VC, or if R be the resistance of the wire between the two points, C^2R.

The activity in the part of the circuit considered is

always VC, but this may be greater than C^2R; in which case work is done otherwise than in heating the conductor. C^2R is then the part of the activity employed in generating heat.

One of the great advantages of the system of units briefly sketched here, is that it states the value of the rate at which work is given out in the circuit, without its being necessary to introduce any coefficient such as would have been necessary if the units had been arbitrarily chosen. When the quantities are measured in C.G.S. units, the value of EC is given in centimetre-dynes, or in *ergs*, per second. Results thus expressed may be reduced to *horse-power* by dividing by the number $7{\cdot}46 \times 10^9$; or if E is measured in volts, and C in amperes, EC may be reduced to horse-power by dividing by 746. Thus, if 90 volts be maintained between the terminals of a pair of incandescent lamps joined in series, and a current of 1·3 ampere flows through these lamps, the rate at which energy is given out in the lamps is approximately ·157 horse-power. If the rate at which work is done in maintaining a current of one ampere through a resistance of one ohm were taken as the practical unit of rate of working, or *activity*, and E reckoned in volts and C in amperes, the rate at which electrical energy is given out in the circuit would be measured simply by EC; and calculations of electrical work would be much simplified. This was proposed by Sir William Siemens (Brit. Assoc. Address, 1882), who suggested that the name *watt* should be given to this unit rate of working. The rate at which energy is given out in the lamps of the above example is $90 \times 1{\cdot}3 = 117$ *watts*. A *watt* is therefore equivalent to 10^7 ergs per second.

The Electrical Congress held at Paris in August 1889

has adopted the watt as the practical electrical unit of work, and the term *kilowatt*, proposed by Mr. W. H. Preece to designate an activity of 1000 watts or 10^{10} ergs per second. It is intended that the latter unit should be used instead of the horse-power. An activity given in kilowatts can be reduced to horse-power by dividing by ·746 or roughly by multiplying by 4 and dividing by 3.

Sir William Siemens also proposed to call the work done in one second, when the rate of working is one watt, one *joule*. Thus the work done in one second in maintaining a current of one ampere through one ohm, or the work obtained by letting down one coulomb of electricity through a difference of potential of one volt, is one *joule*. A *joule* is therefore equivalent to 10^7 ergs, and the work done in one second in the above example is 117 *joules*.

The Electrical Congress of August 1889 has also adopted the joule as the practical unit of electrical work.

CHAPTER VI.

POTENTIALS AND CURRENTS IN DERIVED CIRCUITS.

IN the preceding chapter we have given a short explanation of Ohm's law; in the present chapter we shall consider the consequences of this law a little more at length, and show how to derive from it rules for the calculation of the equivalent resistance of any arrangement of derived circuits, and the strength of the current in any part of that arrangement. Suppose that we have an electric generator E, the two terminals of which are joined by a single copper wire. Then if C be put for the strength of the current flowing in a wire of resistance R between two cross-sections at potentials V_1, V_2 respectively, the results stated at p. 66 above are all expressed, and unit resistance is defined, by the equation

$$C = \frac{V_1 - V_2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Ohm used the expression "Gefälle der Elektricität" for a quantity which, in the earlier works which appeared after the publication of his essay, was called "Difference of Tensions," but which is now recognized as proportional to $V_1 - V_2$; and it is still usual to give a special name to difference of potential when considered in connection

with the flow of electricity. Thus the name *Electromotive Force* is frequently given to the difference of potential between two points or two equipotential surfaces in a homogeneous conductor, when thus considered with reference to flow of electricity from one to the other, and in accordance with custom and authority the term may be thus employed. A somewhat more general sense in which the term is used will presently be explained. It is to be carefully remembered, however, that electromotive force is not a *force:* the two words must be taken *together* as a single term having the meaning assigned to it in its definition.

A constant difference of potential may be maintained between the extremities of a homogeneous conductor, and therefore also a current maintained in the conductor, in several different ways: for example, by a voltaic battery, a thermo-electric pile, or a dynamo-electric or magneto-electric machine. Particulars regarding different forms of voltaic batteries, and the practical construction of other electric generators, are given in various treatises; at present we deal only with principles which are generally applicable, reserving for consideration later their applications in particular cases.

Equation (1) is not fulfilled in general by a conductor made up of different homogeneous portions, put end to end, or by a conductor moving across the lines of force of a magnetic field. For such cases

$$\gamma = \frac{V_1 - V_2}{R} + \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where V_1, V_2 denote as before the potentials at the extremities of the conductor, and R the sum of the resistances of the homogeneous portions of the conductor in

the former case, or the actual resistance of the conductor in the latter. The conductor in such cases is said to *contain*, or to be the *seat of, an electromotive force E*, or (as frequently in what follows) an electromotive force E is said to be *in* the conductor. The total electromotive force producing a current in the conductor is now $V_1 - V_2 + E$. Since in a heterogeneous conductor (1) applies in the first case to every part, except any, however small, which includes a surface of discontinuity, the electromotive force is said to have its seat at the surface or surfaces of discontinuity. In the other case electromotive force has its seat in every part of the conductor moving in the field, according to a law which we shall afterwards discuss.

In a circuit composed of different homogeneous conductors not moving across lines of magnetic force let adjacent points be taken on opposite sides of each surface of continuity, and let the difference of potential between the pair of points in each conductor be measured; the sum of these differences taken in order round the circuit is equal to the sum of the parts of E contributed by the discontinuities. For going round in the direction of the current from a point (not in a surface of discontinuity) to the same point again we have by (2) $V_1 = V_2$ and

$$\gamma = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But denoting the successive homogeneous conductors in their order round the circuit by the suffixes 1, 2 . . . . n, and the differences of potential between their extremities by

$$V_1 - V'_1,\ V_2 - V'_2,\ .\ .\ .\ .\ V_n - V'_n,$$

the corresponding resistances by R_1, R_2, ... R_n, and the total resistance $R_1 + R_2 + \ldots + R_n$ by R, we have

$$\gamma = \frac{V_1 - V'_1}{R_1} = \frac{V_2 - V'_2}{R_2} = \ldots \frac{V_n - V'_n}{R_n} = \frac{\Sigma(V - V')}{R} \quad (4)$$

Hence

$$\Sigma(V - V') = E \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

E is called the *total* electromotive force in the circuit, or simply the electromotive force of the circuit.

We will consider here as an example of the principle just stated its application to the case of a simple voltaic cell composed of two plates of dissimilar metals connected by a liquid, for example, copper and zinc immersed in hydrochloric acid, and connected externally by a copper wire.

Let c and z denote the copper and zinc plates, l the liquid between them. By the theory of the voltaic cell now generally adopted, there is a certain finite difference of potential on the two sides of the junction of the dissimilar metals, and on the two sides of each junction of a metal with the liquid. We may suppose for simplicity the plates to be such that they add no sensible resistance to the circuit, and that therefore the potential may be taken as the same at every point of each. Let V_a denote the potential of the copper plate; V_b the potential of the copper wire close to its junction with the zinc plate; V_{lz} the potential of the stratum of the liquid close to the zinc plate; and V_{lc} the potential of the stratum of the liquid close to the copper plate. The difference of potential between two points in the copper conductor near its ends is therefore $V_a - V_b$, and that between the two sides of the liquid is $V_{lz} - V_{lc}$. Both of these differences are positive, and the current flows from the copper

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plate to the zinc plate through the wire, and from the zinc plate to the copper through the liquid. Further it is an experimental fact, as we shall see later, that the current across any cross-section is the same at every part of the circuit. Calling R the resistance of the copper conductor joining the plates, and r the resistance of the liquid of the cell, we have by (3)

$$\gamma = \frac{V_a - V_b}{R} = \frac{V_{lz} - V_{lc}}{r},$$

and therefore also

$$\gamma = \frac{V_a - V_{lc} + V_{lz} - V_b}{R + r}.$$

But $V_a - V_{lc}$ is the finite difference between the potential of the copper plate and that of the liquid in contact with it, and $V_{lc} - V_b$ is similarly the difference between the potential of the liquid in contact with the zinc plate and that of the extremity of the copper wire adjacent to the zinc plate, and the sum of these two differences constitutes what is called the *Electromotive Force of the cell.* Calling this E, we have

$$\gamma = \frac{E}{R + r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Any other case, however more complicated, might be treated in a similar manner.

If V be the difference of potential between any two points in the copper wire, R the resistance of the wire between these two points, and r the remainder of the resistance in circuit, we have from the equations

$$\gamma - \frac{V}{R} = \frac{E}{R + r},$$

the result

$$V = E\frac{R}{R+r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The activity in the wire is by (2)

$$A = V\gamma = \frac{V^2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and for the whole circuit

$$A = E\gamma = \frac{E^2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

By (8) the activity in any wire not containing an electromotive force can always be found, whatever be the arrangement of which it forms part. The activities in the different parts of more complicated circuits containing Electromotive forces of different kinds will be considered in the chapter on the Measurement of Electric Energy.

If instead of a single cell we have a battery of several cells, its electromotive force is found in exactly the same manner by summing all the finite differences of potential at the surfaces of separation of dissimilar substances in the circuit. Hence if there be n cells in the battery joined in series, that is to say the zinc plate of the first cell joined to the copper plate of the second cell, the zinc plate of the second to the copper plate of the third, and so on to the last cell, the total electromotive force of the arrangement, if the cells have each the electromotive force E, is nE. If the copper plate of the first cell and the zinc plate of the last be joined by a wire, and R denote as before its resistance, r the internal resistance of each cell, a current of strength γ given by the equation

$$\gamma = \frac{nE}{R+nr} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

will flow in the wire. This equation may be written

$$\gamma = \frac{E}{r + \frac{R}{n}},$$

which shows that when n is so great that R/n is small in comparison with r, little is added to the value of γ by further increasing the number of cells in the battery.

The method of joining single cells in series is advantageous when R is large, but when R is comparatively small it fails as shown above, and it is necessary then to diminish r as much as possible. The value of r is, for cells in which, as is generally the case, each plate nearly covers the cross-section of the liquid, nearly in the inverse ratio of the area of the plates, and directly as the distance between them. Hence, by increasing the area of the plates and placing them as close together as possible, the resistance may be diminished. One large cell of small resistance may be formed of several small cells by putting all the copper plates into metallic connection with one another, and similarly all the zinc plates. Several compound cells of large surface thus made may be joined in series. The electromotive force of each compound cell will be E as in a simple cell, but if m cells be joined so as to form one compound cell its resistance will be r/m. If n of these compound cells be joined in series, we have, calling the total external resistance R,

$$\gamma = \frac{nE}{R + n\frac{r}{m}} = \frac{mnE}{mR + nr} \quad . \quad . \quad . \quad (11)$$

If R be not too great, and we have a proper number of

cells, it is possible to arrange the battery so that γ may have a maximum value. There being $m\ n$ cells in the battery the numerator of the above value of γ does not change when the arrangement of cells is varied, and therefore, in order that γ may have its greatest possible value, $m\ R + n\ r$ must be made as small as possible. But identically,

$$m R + n r = (\sqrt{m R} - \sqrt{n r})^2 + 2\sqrt{m n R r}.$$

As the last term on the right-hand side does not vary with the arrangement of the battery, it is plain that $mR + nr$ will have its smallest value when $\sqrt{mR} - \sqrt{nr}$ vanishes, that is when $mR = nr$ or $R = nr/m$, or, in words, when the total external resistance of the circuit is equal to the internal resistance of the battery. It may not be possible in practice so to join a given battery as to fulfil this condition, but if the strongest possible current is required it should be fulfilled as nearly as possible. The method of arranging a battery described in the last paragraph is called joining it in *multiple arc.*

It is to be carefully observed that this theorem applies only to the case in which we have a given battery and have to arrange it so as to produce the *greatest current* through a given external resistance R; and the fallacy is to be avoided of supposing that of two batteries of equal electromotive force, but one having a high, the other a low, resistance, the former is better adapted for working through a high external resistance. Nor is this method of using the battery to be confounded with the most *economical* method. By this arrangement the greatest rate of working in the external part of the circuit is obtained; for by (9) the total rate of working is $nE\gamma$, and the part of this which belongs to the external con-

ductor is $mnE\gamma R/(mR + nr)$, which is a maximum under the same conditions as γ. As much energy is thus given out in the battery itself as in the external resistance, and it is plain that for economy as little as possible of the energy of the battery must be spent in the battery itself, and as much as possible in the working part of the circuit. Hence for economical working the internal resistance of the battery and the resistance of the wires connecting the battery with the working part of the circuit must be made as small as possible. We shall return to this question in Chapter IX.

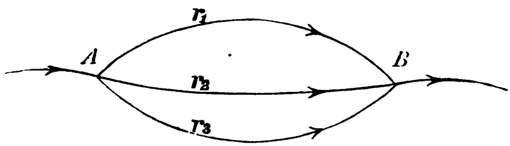

FIG. 9.

We shall now consider shortly the theory of a system of linear conductors (homogeneous wires of uniform section) in which steady currents are flowing.

It has been stated above that when a steady current is kept flowing across any section of a conductor, the current strength is the same across every other section of the conductor; or, in other words, that the flow of electricity at any instant into any portion of the conductor is equal to the flow out of the same portion. This is what is called the *principle of continuity* as applied to the case of a *steady* flow of electricity. By the same principle we have, in the case in which steady currents are maintained in the various parts of a network of conductors, the theorem that the total flow of electricity towards the point at

which several wires meet is equal to the total flow from that point. Thus the current arriving at A (Fig. 9) by the main conductor is equal to the sum of the currents which flow from A by the arcs which connect it with B.

By Ohm's law, if two points A and B, between which a difference of potential V is maintained, be connected by two wires of resistances r_1 and r_2, the current in that of resistance r_1 will be V/r_1 and in the other V/r_2. But if γ be the whole current flowing in the circuit we have by the principle of continuity

$$\gamma = \frac{V}{r_1} + \frac{V}{r_2} = \frac{V}{R},$$

where R is the resistance of a wire which might be substituted for the double arc between A and B without altering the current in the circuit. Hence,

$$\left.\begin{aligned} \frac{1}{r_1} + \frac{1}{r_2} &= \frac{1}{R} \\ \text{and} \qquad R &= \frac{r_1 r_2}{r_1 + r_2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The reciprocal of the resistance R of a wire, that is, $1/R$, is called its *conductance*.* Equation (11) therefore affirms that the conductance of a wire, the substitution of which for r_1 and r_2 between A and B, would not affect the current in the circuit, is equal to the sum of the conductances of the wires r_1 and r_2. From equation (12) we see that the resistance R of this equivalent wire is equal to the product of the resistances of the two wires divided by their sum.

* Formerly called its *conductivity*.

If for r_2 we were to substitute two wires having an equivalent resistance, so that A and B should be connected, as in Fig. 9, by three separate wires of resistances r_1, r_2, r_3, we should have in the same manner for the current in r_1, V/r_1; in r_2, V/r_2; in r_3, V/r_3, and

$$\left.\begin{aligned} \frac{1}{R} &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \\ R &= \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_4} \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

Generally, if two points A and B are connected by a multiple arc consisting of n separate wires, the conductance of the wire equivalent to the multiple arc connection is equal to the sum of the conductances of the n connecting wires; and its resistance is equal to the product of the n resistances divided by the sum of all the *different* products which can be formed from the n resistances by taking $n - 1$ at a time.

As a simple example, we may take the case of a number n of incandescence lamps joined in multiple arc. If the resistance of each lamp and its connections be r, the equivalent resistance between the main conductors, the resistance due to the latter being neglected, is by (11) $r^n/n \cdot r^{n-1} = r/n$. Thus if r be 60 ohms when the lamp is incandescent, and there be twenty lamps, their resistance to the current will be 3 ohms.

By the considerations stated above, we at once deduce from Ohm's law the following important theorem.* In any closed circuit of conductors forming part of any linear system, the sum of the products obtained by mul-

* This theorem and the application of the principle of continuity were first stated explicitly by Kirchhoff, *Pogg. Ann.* Bd. 72, 1847, also *Ges. Abhand.*, p 22.

tiplying the current in each part, taken in order round the circuit by its resistance, is equal to the sum of the electromotive forces in the circuit. This follows at once by an application of Ohm's law to each part of the circuit, exactly as in the investigation in p. 81 above of the electromotive force of the circuit composed of a cell and a single conductor.

As an example of a circuit containing no electromotive forces, consider the circuit formed by the two wires (Fig. 9) of resistances r_1, r_2 joining AB. We

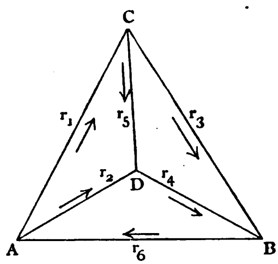

FIG. 10.

have, for the current flowing from A to B through r_1, the value V/r_1; the product of this by r_1 is V; for the current flowing from B to A through r_2 we have $-V/r_2$, and the product of this by r_2 is $-V$: the sum is $V - V$ or zero. As another example, consider the diagram, Fig. 10, of resistances r_1, r_2, r_3, r_4, r_5, between the two points A and B. By what we have seen, if V_a, V_b, V_c, V_d, be the potentials at A, B, C, D, respectively, the current from A to C is $(V_a - V_c)/r_1$, from C to D $(V_c - V_d)/r_5$, and from D to A $(V_d - V_a)/r_2$.

Hence, taking the sum of the products of these current strengths by the corresponding resistances for the circuit $ACDA$, we get

$$V_a - V_c + V_c - V_d + V_d - V_a = 0 \quad . \quad (14)$$

To illustrate the use of the principles which have been established, we may apply them to find the current strength in r_5 (Fig. 10) when r_6 contains a battery of electromotive force E. Let r_6 be the resistance of the battery and the wires connecting it with A and B, and let γ_1, γ_2, &c., be the strengths of the currents flowing in the resistances r_1, r_2, &c., respectively, in the directions indicated by the arrows. By the principle of continuity we get

$$\left.\begin{aligned} \gamma_3 &= \gamma_1 - \gamma_5 \\ \gamma_4 &= \gamma_2 + \gamma_5 \\ \gamma_6 &= \gamma_1 + \gamma_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Applying the second principle to the circuits $BACB$, $ACDA$, $CBDC$, and using equation (14), we obtain the three equations,

$$\left.\begin{aligned} &\gamma_1 (r_1 + r_3 + r_6) + \gamma_2 r_6 - \gamma_5 r_3 &&= E \\ &\gamma_1 r_1 \qquad\qquad - \gamma_2 r_2 + \gamma_5 r_5 &&= 0 \\ &\gamma_1 r_3 \qquad\qquad - \gamma_2 r_4 - \gamma_5 (r_3 + r_4 + r_5) &&= 0 \end{aligned}\right\} (16)$$

Eliminating γ_1 and γ_2, we find

$$\gamma_5 = \frac{E\,(r_2 r_3 - r_1 r_4)}{D} \quad . \quad . \quad . \quad . \quad (17)$$

where

$$D = r_5 r_6 (r_1 + r_2 + r_3 + r_4) + r_5 (r_1 + r_3)(r_2 + r_4) + r_6 (r_1 + r_2)(r_3 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3) \quad . \quad (18)$$

By substituting for γ_2 in the second and third of equations (16) its value $\gamma_6 - \gamma_1$, we get,

$$\left.\begin{array}{l}\gamma_1(r_1 + r_2) + \gamma_5 r_5 \qquad\qquad\qquad - \gamma_6 r_2 = 0 \\ \gamma_1(r_3 + r_4) - \gamma_5(r_3 + r_4 + r_5) - \gamma_6 r_4 = 0\end{array}\right\} \quad . \quad (19)$$

From these we obtain by eliminating γ_1,

$$\gamma_5 = \frac{\gamma_6(r_2 r_3 - r_1 r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \quad . \quad (20)$$

By means of equations (17) and (20) we can very easily solve the problem of finding the equivalent resistance of the system of five resistances r_1, r_2, &c., between A and B. For let R be this equivalent resistance, since γ_6 is the current flowing through the battery, we have $\gamma_6 = E/(r_6 + R)$. Substituting this value of γ_6 in (20), equating the values of γ_5 given by (17) and (20), and solving for R, we get

$$R = \frac{r_5(r_1+r_3)(r_2+r_4) + r_1 r_3(r_2+r_4) + r_2 r_4(r_1+r_3)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \quad (21)$$

It follows from Ohm's law, and the theorems which have been deduced from it, that any two states of a system of conductors may be superimposed; that is, the resulting potential at any point is the sum of the potentials at the point, the current in any conductor the sum of the currents in the conductor, and the electromotive force in any circuit the sum of the electromotive forces in the circuit, in the two states of the system.

The following result, which is a direct inference from the foregoing principles, and can be verified by experiment, will be of use in what immediately follows.*

* For an elementary proof of certain general theorems regarding a network of conductors see the author's *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. i. p. 166.

Any two points in a linear circuit which are at different potentials may be joined by a wire without altering in any way the state of the system, provided the wire contains an electromotive force equal and opposite to the difference of potential between the two points. For the wire before being joined will in consequence of the electromotive force have the same difference of potential between its extremities as there is between the two points, and if the end of the wire which is at the lower potential be joined to the point of lower potential it will have the potential of that point, and no change will take place in the system. The other end will then be at the potential of the other point, and may be supposed coincident with that point, without change in the state of the system. The new system thus obtained plainly satisfies the principle of continuity, and the theorem of p. 88 above, and is therefore possible; and it can be proved that it is the only possible arrangement under the condition that the state of the original system shall remain unaltered.

As a particular case of this result any two points in a linear circuit which are at the same potential may be connected, either directly or by a wire of any resistance, without altering the state of the system.

The following result is easily proved, and is frequently useful. If the potentials at two points A, B, of a linear system of conductors containing any electromotive forces, be V, V' respectively, and R be the equivalent resistance of the system between these two points, then if a wire of resistance, r, be added, joining AB, the current in the wire will be $(V - V')/(R + r)$. In other words the linear system, so far as the production of a current in the added wire is concerned, may be regarded as a single conductor of resistance R connecting the points AB and containing

an electromotive force of amount $V - V'$. For let the points A and B be connected by a wire of resistance r, containing an electromotive force of amount $V - V'$ opposed to the difference of potential between A and B, no current will be produced in the wire, and no change will take place in the system of conductors. Now imagine another state of this latter system of conductors in which an equal and opposite electromotive force acts in the wire between A and B, and there is no electromotive force in any other part of the system. A current of amount $(V - V')/(R + r)$ will flow in the wire. Now let this state be superimposed on the former state, the two electromotive forces in the wire will annul one another, and the current will be unchanged. The potentials at different points, and the currents in different parts, of the system, will be the sum of the corresponding potentials and currents in the two states, and will therefore, in general, differ from those which existed before the addition of the wire.

As an example consider a circuit between two points of which there is a difference of potential V, and let r, r' be the resistances of the two parts of the circuit between the two points. (These two parts may of course be any two networks of conductors joining the points.) Then the equivalent resistance is $rr'/(r + r')$; and if another conductor of resistance R, and not containing any electromotive force, be connected between the two points, the new difference of potential V' will be given by

$$V' = V \frac{R}{R + \dfrac{rr'}{r + r'}},$$

since it has been shown above that V'/R is the current in the conductor.

Hence in order that V' may be approximately equal to V, R must be great in comparison with $rr'/(r + r')$. But $rr'/(r + r')$ can be written in either of the forms $r/(1 + r/r')$, $r'/(1 + r'/r)$, which shows that r and r' are each greater than $rr'/(r + r')$. Hence if R be great in comparison with either of the two resistances r, r , V' will be approximately equal to V, no matter how the electromotive force may be situated in the circuit. This result is useful in connection with the measurement of the difference of potential between two points of a circuit by a galvanometer, as it is only necessary to make the resistance in the circuit of the instrument great in comparison with that of either part of the circuit lying between the two points, to be sure that the difference of potential is practically unaltered by the application of the instrument.

CHAPTER VII.

STANDARD ELECTRICAL MEASURING INSTRUMENTS AND THEIR GRADUATION.

ONE of the most important operations now performed in an electrical laboratory is the standardizing of instruments for the absolute measurement of currents and differences of potential. The process consists in so graduating the scale or indicating arrangement of the apparatus that currents or differences of potential are either read off directly in absolute units, or are so obtained with only a very simple reduction from the reading. One object of this book is to describe methods of standardizing such instruments, and to do this to most advantage it is desirable to first describe an actual instrument of each class, and to use it as an example in describing methods of graduation. The methods will lose nothing in generality by being described in this connection, and will be easily applicable to other instruments.

The instruments adopted for description are Sir William Thomson's well-known Standard Electric Balances, which are now constructed so as to give a very long range of sensibility in the measurement of currents and potentials, his Magnetostatic Current-Meter, and his Quadrant Electrometer and Electrostatic Voltmeters.

The first-named instruments are based on the principle, set forth in Chap. III. above, of the mutual action between the fixed and movable portions of a circuit carrying a current. Each of the mutually influencing portions consists in most of the instruments of one or more complete turns or spires of the conductor, but in some cases consists of only half or part of a turn. In all cases in what follows we shall, following Sir W. Thomson, call each portion a *ring*.

In each of the balances, except that for very strong currents (the kilo-ampere balance), the movable portion of the conductor consists of two rings, carried with their planes horizontal at the extremities of a balance beam free to turn in the ordinary way round a horizontal axis. Above and below each ring on the beam is a fixed ring with its plane parallel to that of the movable ring. The rings are (except in the Composite Balance, p. 107 below) all joined in series, and the current to be measured is sent through them so that the mutual action between the movable ring at one end and each of the two fixed rings there is to raise that movable ring, while the mutual action of the other group of three rings is to depress the corresponding movable ring. The action is therefore to turn the beam round the horizontal axis on which it is pivoted, with for any given position a couple varying as the square of the current flowing.

Fig. 11 shows diagrammatically the rings and the course of the current through them. a, e, b, f are the two pairs of fixed rings, c, d the movable rings. The current entering by the terminal T passes round all the rings in series, in the two movable rings in opposite directions, and returns to the terminal T_1. Since each movable ring is in general in a magnetic field, terrestrial or

artificial, which has a horizontal component, it tends to set itself so that the greatest number of horizontal lines of force may pass through it (Chap. III. above) and therefore is acted on by a couple which tends to turn the beam round its axis. But since the current passes round the movable coils in opposite directions, and these are very approximately equal, the two couples are nearly equal and opposite, and the instrument is practically free from disturbance by horizontal magnetic force.

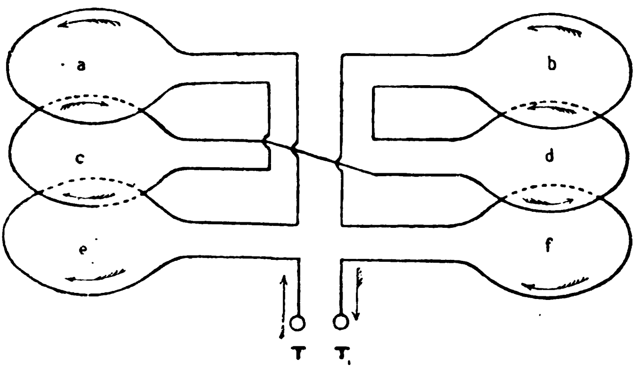

FIG. 11.

The turning couple produced by the mutual action of the fixed and movable rings is balanced for the horizontal or "sighted position" of the beam by an equal and opposite couple produced (in the manner more particularly described below) by a stationary weight at the end of the beam, and a sliding weight placed, steelyard fashion, at a suitable point on a graduated bar attached to the beam. The amount of the current flowing in the rings is de-

duced from the amount of the equilibrating couple thus applied, or rather from a number proportional to it, by the end of a table of reckoning, as described below.

In the kilo-ampere Balance (Figs. 12 and 15) there are only two rings, each nearly rectangular in shape, and horizontal in position. These, since they have to bear very strong currents, are made of massive copper strip. The

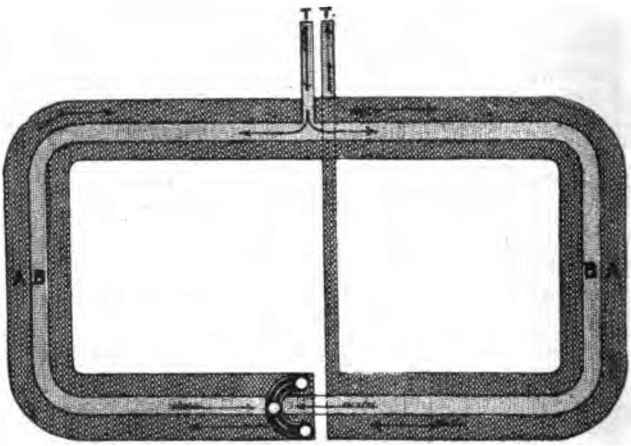

FIG. 12.

movable ring is mounted as shown above the fixed ring, and is free to turn round a horizontal axis halfway between its two ends. In the diagram (Fig. 12) a plan of the fixed and movable rings (*A*, *B*, respectively), with the course of the current in them, is shown. The current enters the movable ring by the terminal *T*, and divides, passing round its two halves in opposite directions. At

the middle of the other side the two parts of the current meet and pass to the fixed ring, then completely round that ring once or more times, and across to the terminal *T*. One half of the movable ring thus tends to rise, the other to fall, and hence the ring tends to turn on its axis. The turning couple is resisted by a return couple due to weights precisely as in the former case.

Having thus stated the general principle of these instruments, we now consider their construction and mode of action with greater minuteness. Most of the constructive details will be made out from Fig. 13 which shows what is called the Standard Centi-ampere Balance, and illustrates the arrangement of the beam, the graduation, and the mode of applying the equilibrating couple, for all the instruments.

The beam in all the instruments is hung on two trunnions, each supported by a flat elastic ligament made of fine copper wires, through which the current passes to and from the movable rings.

The horizontal or sighted position of the beam is that in which the pointers on the extreme right and left are at the middle divisions of their scales. This position, in all the instruments in which a movable ring is acted on by two fixed rings between which it is placed, is not that midway between these two rings, as that would be a position of minimum force and therefore of instability. For stability it is so chosen that the movable ring is nearer to the repelling fixed ring than to the attracting ring by such an amount as to give about $\frac{1}{8}$ per cent. more than the minimum force.

Fixed to the beam and parallel to it is a finely graduated bar, and above this is a horizontal fixed scale, called the Inspectional Scale, less finely divided. Both gradua-

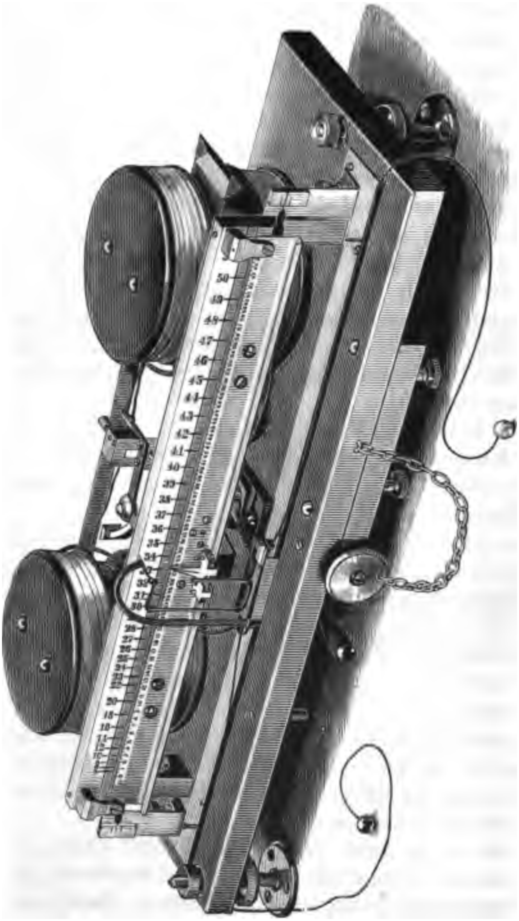

FIG. 13.—STANDARD CENTI-AMPERE BALANCE.

tions begin from zero on the extreme left and have numbers increasing towards the right. A carriage is moved along the graduated bar to any required position by a sliding piece controlled by a cord which can be pulled from either end, and this carriage, by itself or with an additional weight, forms the movable weight referred to above. The position of the carriage is indicated by a pointer which moves along the lower scale. Each additional weight has in it a small hole and slot which pass over conical pins in the carriage. This ensures that the weight is always placed in a definite position. The balancing weight is moved along the beam by means of a self-releasing pendant carried by the sliding piece above referred to. To this pendant is attached a vertical arm (seen in the figure) which passes up through the recess in the front of the weight and carriage and so enables the carriage to be moved with the sliding piece. The stationary weight is placed in the trough shown at the right-hand end of the instrument. The trough is V shaped, and the weight cylindrical, with a cross pin which passes through a hole in the bottom of the trough. The weight is thus placed in a perfectly definite position and always has the same leverage. It is so chosen as just to keep the beam in the sighted position when the sliding weight is at the zero of the scale.

Since the mutual action of the rings is to bring the beam towards the sighted position when displaced by the weights, and the equilibrating couple is that due to the displacement of the sliding weight from zero, the latter couple increases as the current increases, and hence motion of the sliding weight towards the right corresponds to increasing currents. The use of the stationary weight gives a scale of double the length which would be obtained without it.

In the top of the lower or finely graduated scale are notches which correspond to the exact integral divisions in the upper fixed scale. Thus the reading in the fixed scale is got when the pointer is at a notch, without error from parallax due to the position of the eye. The reading when the pointer is between two notches is easily obtained by inspection and estimation with sufficient accuracy for most practical purposes. When however the greatest accuracy is required, the reading is taken on the lower scale, with the aid of a lens, and the current strength calculated from the table of doubled square roots given in the Appendix below.

Four pairs of weights are given with each instrument. Of these one set is for the sliding platform, the other set are the corresponding counterpoises. The weights of each set are in the ratios 1 : 4 : 16 : 64, and are so adjusted that, when the carriage is placed with its index at a division of the inspectional scale, the instrument shows a current of an integral number of amperes, half-amperes, or quarter-amperes, or some decimal subdivision or multiple of one of these units of current.

The accurate adjustment of the zero is effected by a small metal flag as in a chemical balance. This flag is set in any required position by means of a fork moved by a handle beneath and outside the case of the instrument. The sliding weight is brought to zero with the corresponding counterpoise in the trough, and then the flag is turned to one side or the other until the pointer of the beam (seen on the extreme right and left in Fig. 13) is just at zero.

When necessary for transit or otherwise, the beam in the centi-ampere and deci-ampere balances is lifted off its supporting ligament by turning an eccentric by a

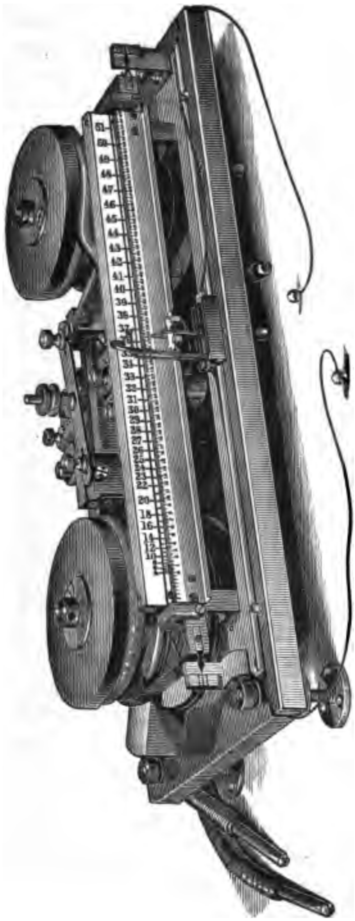

FIG. 14.—STANDARD DEKA-AMPERE BALANCE.

shaft under the sole-plate of the instrument. In the other balances the beam is fixed for carriage by placing distance pieces between the upper and lower parts of the trunnions and screwing them together by milled headed screws kept always in position for the purpose.

Fig. 14 shows the standard deka-ampere balance for the measurement of currents ranging from 1 to 100 amperes. The only essential difference in construction in this instrument consists in the use of a small number of turns of thick copper-wire for the rings, more massive connections to carry the current to and from the movable rings, and special electrodes, so that the instrument can be placed in an electric-light current without perceptibly increasing its resistance.

In this balance (and in the similar hekto-ampere balance adapted for currents from 6 to 600 amperes), when made so as to measure alternating as well as continuous currents, the current is carried by a twisted rope of copper wires, each of which is insulated from its neighbours along its length. The object of this arrangement is to prevent the distribution of an alternating current over the cross-section of the conductor from being affected by inductive action.

Fig. 15 shows the kilo-ampere balance, for currents varying from 25 to 2,500 amperes. Of this we have already described the main parts. The details are similar to those of the other balances, and will be easily made out from the cut. The rings are massive, and the lower fixed ring, shown in the figure as having several turns, is made of bare copper strip, the different turns of which are kept apart by slips of mica.

The centi-ampere balance may be used as a voltmeter

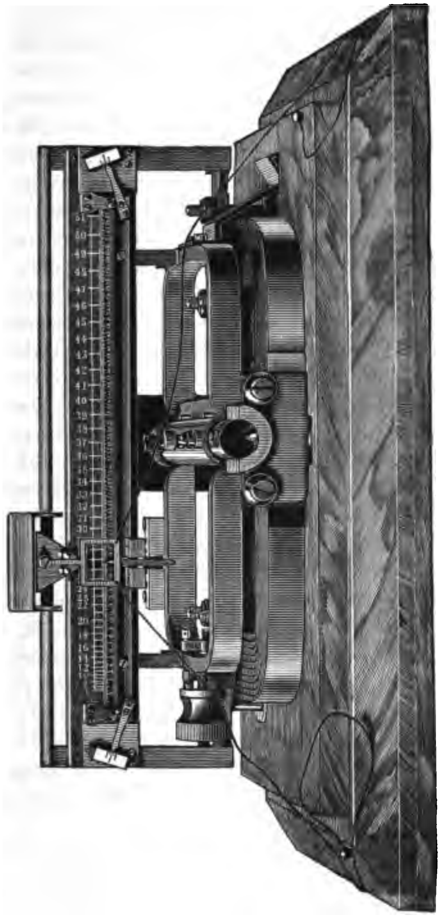

FIG. 15.—STANDARD KILO-AMPERE BALANCE.

by connecting it to two points in a circuit between which the difference of potential is to be ascertained. (On account of heating effects it is not desirable to use in this application of the instrument currents which are greater than can be measured with the lightest weight on the beam, viz. the carriage itself.) According to the difference of potential to be measured, one or more of four platinoid resistances, anti-inductively wound (p. 188 below) provided with the instrument, is to be included in series with the instrument between the two points in question. One of these with the resistance of the coils of the instrument just makes up 500 ohms at a certain specified temperature; each of the others has a resistance of 500 ohms at that temperature. The smallest of these resistances is used when the smallest differences of potential are being measured, and in other cases one or more of the others as may be required to bring the current passing through the instrument down to one which can be measured with the smallest weight on the beam.

Of course deviation of the temperatures of the coils and the resistance employed, produced by the current or otherwise, from the temperatures at which they are correct, must be determined and allowed for in this application of the instrument. The correction is, for the copper coils of the balance ·38 per cent. per 1° C., and for the platinoid resistance coils about ·024 per cent. per 1° C. The temperatures within the coils and the resistances are found by inserting the bulb of a thermometer sent out with the instrument into orifices provided for the purpose.

The difference of potential in volts, V, between the points at which the terminals of the derived circuit, formed by the balance and coils, are applied, is obtained

from the current in amperes, C, and the resistance in ohms, R, by the equation

$$V = C R.$$

The value of R is, as stated above, either 500, 1,000, 1,500, or 2,000, and hence the reduction from the value of C can be made at once mentally without risk of error. If R be great in comparison with the resistance of the part of the circuit included between the terminals, this value of V may of course be taken as that of the difference of potential existing between these points when the instrument is not applied.

At present this application may be considered as made only for continuous currents. The subject of alternating currents will be considered in Chap. IX.

A form of balance called a Standard Composite Balance (Fig. 16) has been constructed for the measurement of the current flowing in a circuit, or the difference of potential between two points, or the product $V C$ of these two quantities expressed in amperes and volts respectively, that is the activity in watts (see p. 76 above) between the two points considered.

The details of the beam, scales, &c., are similar to those of the balances already described, but the arrangement of coils is different. This is shown in the diagrammatic sketch (Fig. 17). Each coil of the fixed pair at one end of the beam is made of thick wire represented by the heavy lines on the right-hand side. These coils have a separate pair of electrodes, E, E_1. The coils of the pair at the other end, and the coils on the beam, are of fine wire, as in the centi-ampere balance, and can be joined in the manner described below with a pair of electrodes T, T_1.

The switch in the diagram enables the fine wire coils

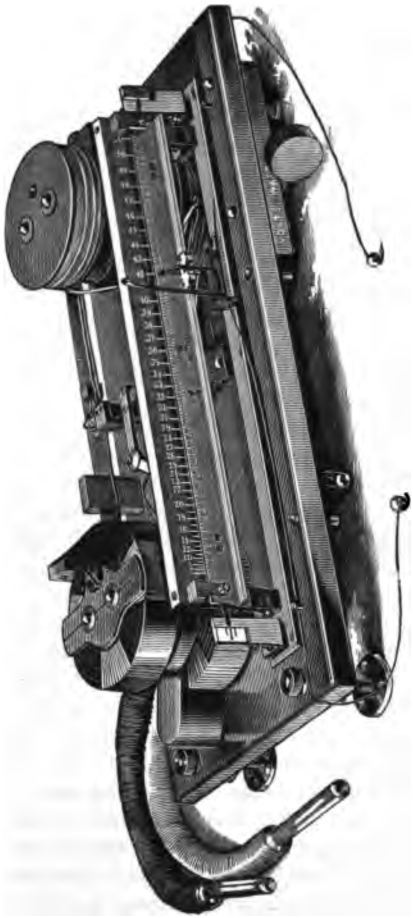

FIG. 16.—COMPOSITE BALANCE.

on the beam to be joined in series with the fixed fine wire coils and connected with T, T_1, or to be connected with the terminals T, T_1, by themselves while the fixed fine wire coils are left out of circuit. The former arrangement has place when the switch is turned to *watts*, the latter when the switch is turned to *volts*. The actual switch with these directing words will be readily seen in Fig. 16.

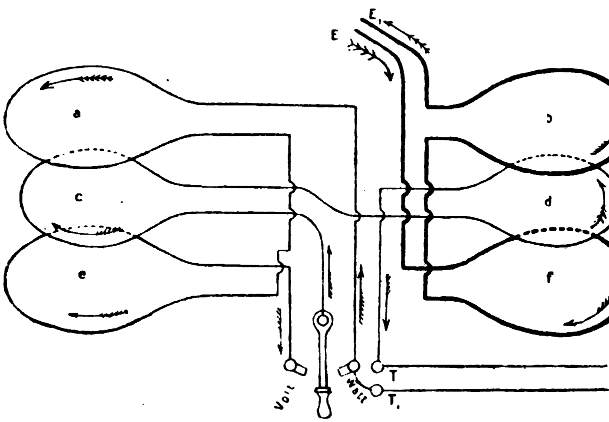

FIG. 17.

When the switch is turned to *watts* the instrument is joined for use as an activity-meter. The thick wire coils are included in the circuit, so that the main current flows through these, while the movable coils are placed by the terminals T, T_1 so as to connect in series with a suitable anti-inductive resistance, sent out with the instrument, two points of the circuit through which the current C in the thick wire coils is flowing. The action between

the fixed and movable coils is then directly as the product of V and C. These quantities are to be found as follows, and $V\,C$ calculated from their values.

To find V the instrument is arranged for use as a voltmeter or centi-ampere meter by connecting the switch to *volts*, and connecting the instrument through a suitable resistance to the two required points. The fixed fine wire coils and the movable coils at one end thus act mutually on one another. One or other of three special weights marked $V.W_1$, $V.W_2$, $V.W_3$, is used, and the current calculated from the constant given in the certificate sent with the instrument. The difference of potential is then found in volts by multiplying this by the resistance in the circuit of the instrument taken in ohms.

The instrument may also be used as a hekto-ampere meter and the value of C found. The switch is turned to *watts*, and the thick wire coils so joined by their electrodes in the circuit that the movable coil at that end is repelled up. A measured current is then sent through the movable coils. This current, as used in the graduation of the instrument, is ·25 ampere, but any other current which is convenient and not too great may be used. Either the carriage alone or the special weight marked $W.W$ is to be used in bringing the beam to the sighted position. If the measured current in the fine wire coils be c the current C in the thick wire coils will be obtained in amperes by multiplying the constant given in the certificate for the reading by $4c$.

The current in the thin wire coils may be measured by the fixed and movable thin wire coils arranged as for the measurement of volts. The switch is turned to *volts*, and the terminals, including the instrument and a suitable anti-inductive resistance, are connected with

those of an electric installation or battery and the current measured; then the switch is turned to *watts* and a resistance added to the circuit equal to that of the fixed fine wire coils which has been withdrawn. The measurement is then made as described above.

If V be the potential in volts, R the resistance included in ohms, c the measured current in amperes sent through the fine wire coils, A the activity in watts, we have

$$A = VC = cCR.$$

The following is a list of the Standard Balances with their ranges, and a table showing for each instrument as numbered, the current per division of the inspectional scale corresponding to each of the four pairs of weights.

Name of Balance.	Range.
I. Centi-ampere Balance.	1 to 100 Centi-amperes.
II. Deci-ampere Balance.	1 to 100 Deci-amperes.
III. Deka-ampere Balance.	1 to 100 Amperes.
IV. Hekto-ampere Balance.	6 to 600 Amperes.
V. Kilo-ampere Balance.	25 to 2,500 Amperes.
VI. Composite Balance.	·02 to 300 Amperes. 100 to 25,000 Watts (at 100 volts.)

	Current for each division of inspectional scale for the different balances.				
	I. Centi-amperes per division.	II. Deci-amperes per division.	III. Amperes per division.	IV. Amperes per division.	V. Amperes per division.
First pair of Weights.........	·25	·25	·25	1·3	5
Second ,, ,, ...	·50	·50	·50	3·0	10
Third ,, ,, ...	1·0	1·0	1·0	6·0	20
Fourth ,, ,, ...	2·0	2·0	2·0	12·0	50

Another instrument of Sir William Thomson's may be taken as typical of the class of instruments on the ordinary galvanometer principle in which the sensibility is varied by altering the directive force of the stationary magnetic field at the needle. This instrument, which he has called a *magnetostatic current-meter*, has from its mode of construction a very wide range of sensibility for practical work.

The current traverses two coils each consisting of one or more turns of copper conductor, and produces a uniform magnetic field in part of the space between them. In that space is hung the movable system of small steel magnets or "needle."

The coils are enclosed within a brass cylinder supported vertically on three levelling feet, and carrying on its upper end a shallow circular scale-box with glass roof. An index, supported by a jewelled cap on an iridium point, plays round the scale, which is graduated so that the reading is proportional to the tangent of the angular deflection, and has 100 divisions. A vertical shaft carried by the index, and passing down through the bottom of the scale box supports the needle.

Two systems of directive magnets produce the stationary field, and consist of two rings of magnetized steel, round the case of the instrument in horizontal planes, one above, the other below the plane containing the magnetic axis of the coils. Each ring is made of two semicircular magnets of rectangular section placed with their similar poles together in an annular brass frame fitted on the cylindrical case so that the rings can be turned either together or separately about a vertical axis through the centre of the suspended system of magnets. Thus, by altering the positions of the rings, the directive force on the needle may be

varied from a maximum when the rings are similarly placed as to their poles to a minimum when they are oppositely placed, and the instrument can therefore be adapted to measure currents widely differing in strength. It has also the advantage of permitting its constant to be varied (when a standard instrument is available) to any value greater than one-tenth of that for the most powerful directive field.

When the instrument is moved from place to place, its index is raised off the iridium point by means of a ring-shaped lifter below it, which can also be used to check the vibrations of the index and suspended needle.

The instrument is made in two grades of sensibility, the Milliamperemeter, of effective range ·3 to 300 milliamperes, reading 2 milliamperes per division of the scale, and the Amperemeter, of range ·3 to 300 amperes, reading 1 ampere per division.

The Milliamperemeter like the centiampere balance (p. 104) can be used conveniently as a voltmeter, and the method of so using it is similar to that already described. Two platinoid resistances, wound anti-inductively on a copper cylinder, are supplied to be used in series with the coils of the instrument. The resistance of the coils (about 40 ohms of copper wire) makes up with one of these resistances 100 ohms; the other is 900 ohms. Thus if the instrument reads 2 milliamperes per division of the scale in the ordinary use, it will give, with the smaller resistance in series with the coils, $\frac{1}{5}$ of a volt per division; with both resistances in series with the coils 2 volts per division.

Differences of potential are also frequently measured by electrostatic instruments. These take several different forms, but each consists essentially of two conductors

between which is established the difference of potential which it is desired to measure. The electrostatic force set up produces motion of the parts of one of these conductors relatively to one another, or motion of the conductor as a whole relatively to the other conductor, and from the displacement thus produced, or from the mechanical force which must be applied to restore and maintain equilibrium in the configuration of zero electrification, the difference of potential is inferred.

The general principle on which an absolute electrometer has been constructed by Sir William Thomson is stated above, and a full description of the instrument will be found in his *Reprint of Papers on Electrostatics and Magnetism*, or the author's *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. i. Here we shall only describe three electrostatic instruments—Sir William Thomson's Quadrant Electrometer and Electrostatic Voltmeters. These are not absolute instruments in the sense of containing within themselves a means of comparing the electric force called into play with other forces known in amount, as for example the force of gravity on a given mass, or the elastic reaction of a stretched spring. The indications of such instruments can only be obtained in absolute measure by a comparison with those of an absolute electrometer or some form of standard voltmeter, as for example the standard centiampere balance described above used as a voltmeter.

The Quadrant Electrometer had its beginning in the Divided Ring instrument illustrated in Fig. 18. A vertical wire carrying on one side a light horizontal needle is suspended from a fixed point. The wire passes through the centre of two flat semicircular pieces of metal, which

lie in a horizontal plane so as to form a metallic circle complete with the exception of a small space at each extremity of a diameter. These spaces insulate one semicircle from the other. Supposing the needle charged with positive electricity and made to rest in equilibrium above one of these spaces when the two semicircles are put in conducting contact, the arrange-

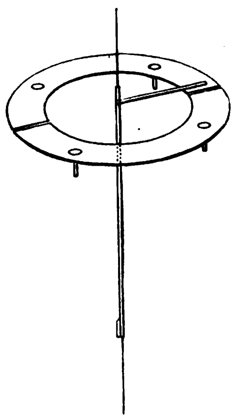

FIG. 18.

ment is symmetrical about the needle. If one semicircle be then charged with positive the other with negative electricity, the needle will be repelled from the positive and attracted toward the negative semicircle. If then the wire be brought back and maintained in the symmetrical position by an applied couple, this couple gives a measure of that due to electric forces tending to deflect the needle, and if the potential of the needle

remains constant, differences of potential established between the semicircles can be compared.

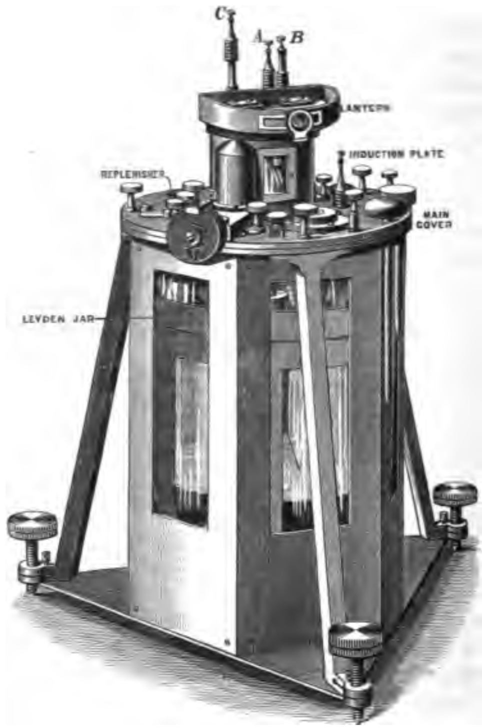

FIG. 19

It was an obvious but important step to convert the two semicircles into four quadrants by a pair of openings along a diameter at right-angles to the other pair, to put

each pair of opposite quadrants into conducting contact, and to make the needle symmetrical about the suspension wire. Thus supposing one pair of quadrants to be charged positively and the other pair negatively, one end of the needle is attracted by one of a pair of quadrants, and repelled by the adjacent quadrant of the other pair. The other end of the needle is attracted by the remaining quadrant of the first pair, and repelled by the remaining quadrant of the other pair, which is adjacent. These

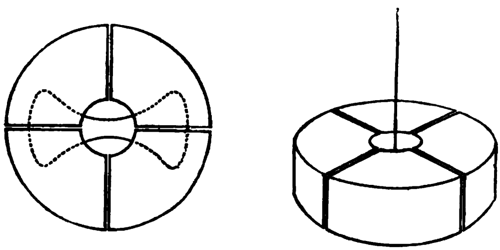

FIG. 20.

actions conspire to give a couple turning the needle about the suspension wire.

In the final form of the quadrant electrometer, which is represented in Fig. 19,* the four quadrants of the flat-ring are replaced by four quadrants of a flat cylindrical box made of brass. These are shown separately in Fig. 20. Each quadrant is supported on a glass stem projecting downwards from a brass plate which forms the cover of a Leyden jar, within which the quadrants and needle are enclosed. For three of the quadrants the stem

* Figs. 19 and 21 are taken by permission from Messrs. Stewart and Gee's *Practical Physics*, vol. ii.

passes through a slot in the cover and is attached to a brass piece which closes the slot from above. Thus each of the quadrants can be moved out or in through a small space. The stem of the fourth quadrant is attached to a piece above the cover which rests on three feet. Two of these feet are kept by a spring in a V-groove, parallel to which the piece carrying the quadrant with it can be moved by a micrometer-screw turning in a nut fixed to the movable piece. The spring which keeps the feet of the movable piece in their groove presses outwards as well as downwards, and so keeps the same sides of the nut and screw threads in contact, to the prevention of "lost time." The details of the instrument will be easily made out by means of Figs. 21 and 22. The former shows a vertical section of the instrument, the latter the suspension-piece and mirror.

A plate rather less in area than the upper surface of a quadrant, but of nearly the same shape, is supported by a glass stem from the cover above a quadrant adjacent to that attached to the micrometer, and is furnished with an insulated electrode passing through the cover. Sufficient length is given to the insulating stem by attaching it to the roof of a cylinder, closed at the top, erected over an opening in the cover. This plate is called the *induction plate.*

Within the box formed by the quadrants and about midway between the top and bottom, a needle of sheet aluminium of the form shown by the line drawn, partly full, partly dotted, across the plan of the quadrants on the left in Fig. 20 is suspended horizontally from two pins, *c*, *d* (Fig. 22), carried by a fixed vertical brass plate supported on a glass stem projecting above the cover of the jar. The needle is attached rigidly at its

centre to the lower end of a stiff vertical wire of aluminium, which passes down through an opening in the middle of the cover.

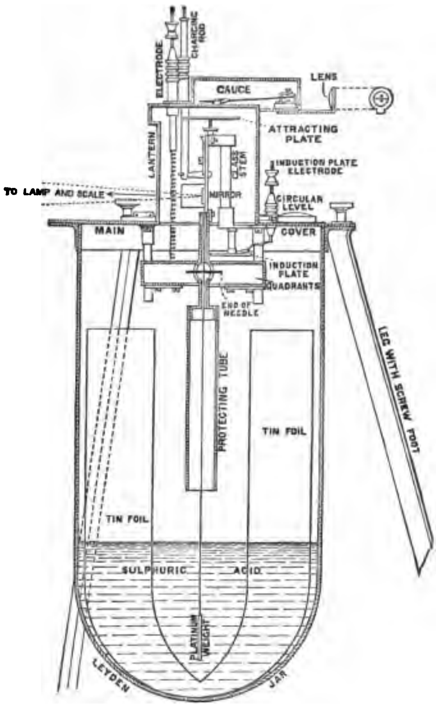

FIG. 21.

To the extremities of a small cross-bar at the top of the aluminium wire are attached the lower threads of a bifilar made of two single silk-fibres. The upper

ends of these fibres are wound in opposite directions round the pins *c*, *d*, each of which has, in its outer end, a square hole to receive a small key, by which it can be turned round in its socket so as to wind up or let

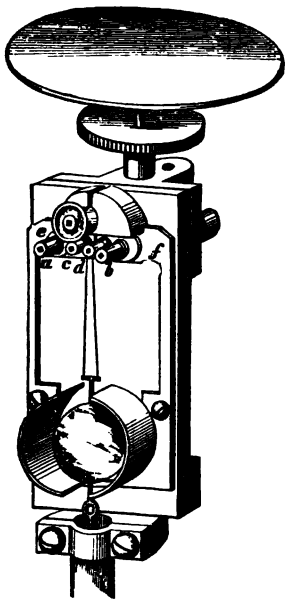

Fig. 22.

down the fibre. By this means the fibres can be adjusted so as to be as nearly as may be the same length; and as the whole supported mass of needle, &c., is then symmetrical about the line midway between the fibres, each bears half the whole weight. The pins *c*, *d*,

are carried by the upper ends *e*, *f*, of two spring pieces which form the continuations of a lower plate screwed firmly to the supporting piece. Through *e*, *f*, and working in them, pass two screws *a* and *b*, the points of which bear on the brass supporting plate behind. By the screw *a* the end *e* of the plate *e f*, can be moved forward or back through a certain range, and thus the pin *c* carried forward or back relatively to *d*; similarly *d* can be moved by the screw *b*. Thus the position of the needle in azimuth can be adjusted. The distance of the fibres apart can be changed by screwing out or in a conical plug shown between the springs *e*, *f*.

The aluminium wire carries between its upper end and the needle a small concave mirror of silvered glass, to be used with a lamp and scale to show the position of the needle. The mirror is guarded against external electric influence by two projecting brass pieces, which form nearly a complete cylinder round it. The part of the wire just above the needle is protected by the tube shown at the bottom of Fig. 22. This tube extends down below the needle a little distance, and is cut away at each side to allow the needle free play to turn round.

The interior coating of the Leyden jar is formed by a quantity of sulphuric acid which it contains, and which also serves to preserve a dry atmosphere within the jar, the exterior coating by strips of tinfoil pasted on its outer surface. The acid has been boiled with sulphate of ammonia to free it from volatile impurities which might attack the metal parts of the instrument. The jar itself is enclosed within a strong metal case of octagonal form, supported on three feet, with levelling screws. The line joining two of these feet (which are in front) is, when level, parallel to the axis of the needle if the latter is properly adjusted.

The needle is connected with the inner coating of the jar by a thin platinum wire kept stretched by a platinum weight at its lower end, which hangs in the acid. The wire is protected from electrical influence by a guard-tube forming a continuation of the narrower guard-tube, (partly shown in Fig. 22) and therefore extending from below the quadrants to a short distance above the acid, and connected also by a platinum wire with the acid.

The supporting plate in Fig. 22 carries the disc of an idiostatic* gauge of the kind described in p. 123 below. The height of the disc is adjustable by means of a fine screw and jam-nut below it. The supporting plate, with the suspension and disc of the gauge, &c., is enclosed within an upper brass case, called the *lantern*, which closes tightly the central opening of the cover. The top of the lantern is the guard-plate of the gauge, and carries the aluminium trap-door and lever with sighting plate and lens as already described.

A glass window in the lantern allows light to pass to the mirror, and the suspension to be seen. A small opening in the glass, closed when not in use by a screw-plug of vulcanite, enables the operator to adjust the suspension without removing the lantern.

The principal electrodes of the quadrants are brass rods cased in vulcanite, and are arranged so as to be movable vertically. Each is terminated above in a small brass binding screw, and is connected below by a light spiral spring of platinum with a platinized contact piece, which rests by its own weight on a part of the upper surface of the quadrant, also platinized to ensure good contact.

* An electrometer in which an auxiliary electrification, independent of that being tested, is employed (as in the ordinary mode of using the quadrant electrometer) is said by Sir William Thomson to be *heterostatic*; if the electrification being tested is alone employed it is said to be *idiostatic*.

They are placed one on each side and in front of the mirror. One is in contact with the quadrant connected below to the micrometer quadrant, the other to the quadrant connected to that below the induction plate.

An insulated charging-rod descends through the lantern, and carries at its lower end a projecting spring of brass. When the rod is not in use the spring is not in contact with anything; but when the jar is to be charged the rod is turned round until the spring is brought into contact with the supporting-plate, which, as stated above, is in contact with the acid of the jar.

The difference of potential between the inner and outer coatings of the jar is tested by an auxiliary attracted disc electrometer used idiostatically. This electrometer, which is called the *gauge*, is contained within the cylindrical box *J* on the cover of the jar. The arrangement is shown in detail in Fig. 23. The disc is a square piece of aluminium forming a continuation of a lever *h* of the same metal. This lever is forked and the prongs joined by a black opaque hair which moves in front of a white enamelled plate on which are two black dots in a vertical line. The position of the hair is seen through the plano-convex lens *l*, which is carried by a sliding platform attached to the guard-ring *G*. Torsion given to the platinum wire *f*, to which the lever is attached in the manner shown in Fig. 23, and round which the lever turns as a fulcrum, forces the disc upwards. This upward force is balanced when the hair is just between the dots by electric attraction between the disc and a parallel plate below it, which is in contact with the interior coating of the jar, while the guard-ring and disc are in contact with the exterior coating. The attracting plate below is mounted on a fine screw, by which its distance from the disc and therefore

the sensibility of the gauge can be varied at pleasure within certain limits. The sensibility of the gauge varies with any alteration in the elasticity of the torsional spring *f*. This however is of little consequence as the variations are not sudden, and it is never necessary to know the actual potential of the jar.

Between each end of the wire *f* and the supporting block is interposed a *U* shaped spring (not shown in Fig. 23) made of light brass. The end of the wire is attached to the extremity of one limb of the *U*, a pin passing through the supporting block to the extremity of the other limb.

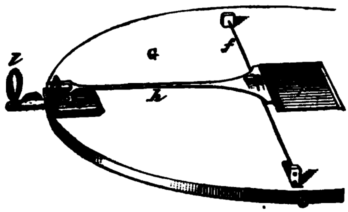

FIG. 23.

The two pins, the extremities of the springs, and the wire are in line. The springs can be turned round the pins as axes, so as to give any initial torsional couple to the wire which may be required, and by their elastic reaction cause the wire to be stretched with a nearly constant force.

The mode of attachment of the wire to the lever *h*, deserves notice. The wire is threaded through two holes in the broader part of the lever, near the square disc, so that the part between the holes is above the lever. Half-way between the holes it passes over a small ridge piece of aluminium, which prevents the lever from turning round without twisting the wire.

The difference of potential between the coatings of the jar is kept nearly constant by means of a small induction machine R, called by Sir William Thomson the ***Replenisher***. The construction and action of this part will be easily

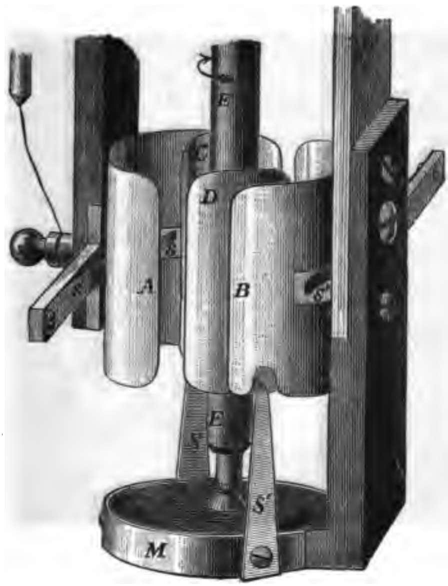

FIG. 24.

understood from Figs. 24 and 25;* Fig. 24 shows the mechanism full-size in perspective elevation; Fig. 25, the same in plan.

* These cuts are taken with permission from Prof. Ayrton's *Practical Electricity*. (Cassell & Co., London.)

Two similar metal carriers C, D, each part of a cylindrical surface, are fixed on a cross-bar of paraffined ebonite so as to be slightly noncoaxial but inclined at the same angle to the cross-bar. Through the cross-bar and rigidly fixed to it, passes a cylinder of ebonite having at

FIG. 25.

its ends metal pieces which form the extremities of an axle. The carriers turn round this axle within the circular cylinder marked out by the cylindrical metallic pieces A, B, which are insulated from one another and act as inductors. A receiving spring, s or s', projects obliquely inwards through a hole in each inductor, with which it is

also connected at the back, and is bent so that the carriers touch the springs on their convex sides, and pass on but little impeded by the friction. Two contact springs S, S', connected by a metallic arc, project slightly inwards beyond the inductors so that the carriers, while opposite the inductors, come into contact with these two springs at the same time, and are therefore put into conducting contact. One of the inductors, A, is connected to the inner coating of the jar, the other, B, is attached by the supporting plate of brass to the cover of the instrument and therefore to the outer coating. A milled head attached to R projects above the cover and forms a handle by which the carriers are twirled round.

The electrical action is easily understood. An initial charge has been given to the jar, so that a difference of potential exists between the coatings, the interior for example being positive. When the carriers come into contact with the springs S, S', they are brought to the same potential, and, since they are under the influence of the inductors, one carrier becomes charged positively, the other negatively. Then, turning in the direction of the arrow, they come into contact with the receiving springs, and being each (electrically) well under cover of the corresponding inductor, they give up the greater part of their charges, thus increasing the difference of potential between the inductors.

If the carriers are turned in the opposite direction the action is of course reversed, and the difference of potential is diminished.

A spring catch keeps the knob of the replenisher, which is on the upper side of the cover, in such a position when not in use that the carriers are not in contact with any of the springs.

On the upper side of the cover of the jar are screws, three in number, securing the cover to a tightly fitting flat ring-collar below it, to which the jar is cemented, and to which the case is screwed; two screws, one on each side, which fix the lantern in its place; a cap covering an orifice communicating with the interior of the jar; two binding screws by which wires can be connected to the case; and a knob similar to that of the replenisher, which, when turned against a stop marked "contact," connects by an interior spring the quadrant below the induction plate with the case, and when turned in the opposite direction to an adjoining stop marked "no contact," insulates that quadrant from the case. Two keys, for turning the pins *a*, *b*, *c*, &c., are kept let down outside the case through holes in the projecting edge of the cover. The cover also carries a small circular level, set so as to have its bubble at the centre when the cover is levelled by an ordinary level. When this has been done the accuracy of construction of the quadrants ensures that they are also level. The level has a slightly convex bottom, and is screwed down with three screws, so that when the instrument is set up for use, a final adjustment, to show horizontality of the quadrants, can easily be made by turning the screws.

Full instructions for setting up and adjusting the quadrant electrometer are sent out with each instrument by the maker, and are therefore available, if kept, as they ought to be, beside it in the case. We shall suppose therefore that the detached parts have been put into their places, the acid poured into the jar, and the instrument set up and levelled; but as a quadrant electrometer is now part of every well-equipped physical laboratory, and is used over a wide range of electrical work, we shall describe here the principal adjustments.

The two front quadrants are pulled out as far as possible, to allow the operator to observe the position of the needle, which should rest with its plane horizontal and midway between the upper and under surfaces of the quadrants. If it requires to be raised or lowered, the operator winds or unwinds the fibres by turning the pins *c*, *d*, to which they are attached. The suspension wire of the needle should pass through the centre of the circular orifice formed in the upper surface of the quadrants, when these are symmetrically arranged. If the wire is not in this position the pins *a*, *b*, are turned so as to carry the point of suspension forward or back until the wire is adjusted, and then one pin is carried forward and the other back, without altering the position of the wire, until the black line along the needle is parallel to the transverse slit separating the quadrants.

The scale is placed at the proper distance to give a distinct image of the wire across the line of divisions in front of the lamp flame, then levelled and adjusted so that, when the image is at rest in the centre, the extremities of the scale are at equal distances from the needle.

When the best relative positions of the instrument and the stand for the lamp and scale have been ascertained, these are fixed by the "hole, slot, and plane" arrangement (p. 7) adopted by Sir William Thomson, to allow any instrument supported on three feet or levelling screws to be removed at pleasure, and replaced without readjustment in its original position. A conical hollow, or better, a hole shaped like an inverted triangular pyramid, is cut in the table so as to receive the point (which should be well rounded) of one of the levelling screws, without allowing it to touch the bottom. A V-groove, with its axis in line

with the hollow, is cut for the rounded point of another levelling screw, and the third rests on the plane surface of the table. If it is desired to insulate the electrometer case it is supported on three blocks of vulcanite cemented to the table; and in one of these the hollow is cut, in another the V-groove.

When the jar is being charged, the main electrodes, the induction plate electrode, and one of the binding screws on the cover, are kept connected by a piece of fine brass or copper wire. The charging electrode is turned round so as to bring the spring at its lower end into contact with the supporting brass piece, and a positive charge is given to the jar by means of the small electrophorus which accompanies the instrument. The cover of the jar is tapped during the process to release the balance lever from the stop, to which it may be adhering. When the lever rises the charging rod is turned so as to disconnect the spring, and the charge is then adjusted to the normal amount (determined by the distance of the attracting disc from the trap door) by the replenisher.

The spot of light may in the process of charging have moved from its position for no electrification, and must be brought back by moving out or in the quadrant carried by the micrometer screw.

In ordinary circumstances the leakage of the jar will cause the hair to fall down in twenty-four hours about half the breadth of the lower black spot. This loss of charge from the jar is made good by the replenisher; but if the leakage is considerably greater, the main stem should be washed by means of a piece of hard silk ribbon (to avoid shreds) with soap and water, then with clean water, and finally carefully dried. Shreds and dust on the needle and quadrants may tend to discharge the jar,

and anything of this kind should be removed by carefully and lightly dusting the needle and quadrants with a clean camel's-hair brush. The jar is selected for its high insulating power, but if the acid has in careless handling of the instrument been splashed over the interior surface there may be considerable leakage over the surface of the jar to the case. This can be remedied by removing the acid and carefully washing the jar. The replenisher may also cause leakage of the jar through a deterioration of insulating power of the vulcanite sole-plate which connects the inductors. Such a deterioration with lapse of time is not uncommon in ebonite, and is a consequence of slow chemical action at the surface. A nearly complete cure can be effected by removing the piece and washing it carefully by prolonged immersion in boiling water, and then re-covering its surface with a film of paraffin.

The insulation of the quadrants is now tested. One pair of quadrants is connected to the case and a charge producing a difference of potential exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected to the electrometer case, while the other is connected for an instant to the electrode of the insulated quadrants; and the deflection of the spot of light is read off. The percentage fall of potential produced in thirty minutes or an hour is obtained merely by taking the ratio of the diminution of deflection which has taken place in the interval to the original deflection. If this is inappreciable the quadrants insulate satisfactorily. In any case, for satisfactory working the rate of loss of potential shown by the instrument should not be greater than that of the body tested.

If the insulation is imperfect the glass stems supporting

the quadrants should be washed by passing round them a piece of hard silk ribbon well moistened and soaped, then with clean water to remove the soap, and dried by a clean piece of silk ribbon well dried and warmed. If this does not succeed, the fault probably lies in the vulcanite insulators of the electrodes, which should be well steeped in boiling water, then re-covered with clean paraffin and replaced. Care must be taken when this is done not to bend the electrodes.

The final adjustment of the tension of the threads to equality is now made. One pair of quadrants is connected to the case, and the other pair insulated. The poles of a single Daniell's cell are then connected to the electrodes, and the extreme range of deflection produced by reversing the battery, either by hand or by a convenient reversing key, is observed. One side of the instrument is then raised by screwing up that side by one or two turns of one of the front pair of levelling screws, and the range of deflection again noted. If the range is greater the fibre on that side is too short, if the range is smaller the fibre is too long; and the length must be corrected by turning one or other of the pins to which the fibres are suspended. The pins can be reached by the aperture in the window of the lantern ordinarily closed by the vulcanite plug; and to prevent discharge of the jar the key with vulcanite handle should be used to turn them. The black line on the needle will require readjustment by the screws after each alteration of the suspension.

Messrs. Ayrton, Perry, and Sumpner have found (*Phil. Trans.*, A., 1890) that the mutual action of the needle and guard-tube makes equation (1), p. 134, inaccurate when the difference of potential between needle and case is much over 100 volts, unless the quadrants have a certain distance

apart (in their instrument 3·9 mms.). Ordinarily however the instrument is used heterostatically, that is, the needle is kept by the jar at a potential much above any measured.

The following is the elementary theory of the instrument. Consider three conductors maintained at different potentials, and fulfilling the following conditions :—One of the conductors (A) (in the quadrant electrometer the needle) is symmetrically placed with reference to the other two (B and C) (in the electrometer the two pairs of quadrants), and is so formed that one of its two ends or bounding edges is well under cover of B, and the other end or edge under cover of C, so that the electric distribution near each end or edge is uninfluenced except by

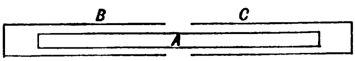

FIG. 26.

the near conductor. One such simple symmetrical arrangement is shown in the figure. Let the potentials of A, B, C be respectively V, V_1, V_2; and let A be slightly displaced from B towards C. This displacement may be angular or linear, according to the arrangement adopted; in the quadrant electrometer it is measured by the angle through which the needle is turned. Let θ denote the displacement and k the electrostatic capacity of A per unit of θ at places not near the ends or bounding edge of A, and well under cover of B and C. Then the quantity of electricity lost by A in consequence of its displacement relatively to B is $k\theta(V - V_1)$, and the quantity lost by B is $k\theta(V_1 - V)$. Similarly the quantities gained by A and C in consequence of the motion

of A towards C are respectively $k\theta(V - V_2)$ and $k\theta(V_2 - V)$. Multiplying the first and second of these quantities by V and V_1 respectively, the third and fourth similarly by V and V_2, subtracting the sum of the two first products from the sum of the second two, and dividing by 2, we get for the work done by electrical forces in the displacement the value

$$k\theta(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right).$$

But this must be equal to the average couple multiplied into the displacement if the latter is angular, or the average force into the displacement if the latter is linear. We have therefore, denoting the force or couple by F,

$$F = k(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right). \quad . \quad . \quad (1)$$

In an arrangement of this kind when the displacement is small the couple or force acting on A is nearly the same over the whole displacement, and thus is nearly equal to the equilibrating couple or force due to the torsion wire, or bifilar, or other arrangement producing equilibrium. But for small displacements this will generally be proportional to the displacement, and therefore also to the deflection D on the scale of the instrument, and thus

$$D = m\theta = c(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad (2)$$

where c is a constant depending on the instrument and the mode of reckoning of D. If V be, as it usually is, great in comparison with V_1 or V_2, then

$$V_1 - V_2 = C'D \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $1/C'$ is the now practically constant value of $c\{V - (V_1 + V_2)/2\}$.

If the angle of deflection θ of the ray of light is not a very small angle, the couple given by the bifilar, it is to be remembered, is proportional to $\sin \frac{1}{2}\theta$. Hence if D be the distance in divisions on the scale (supposed straight and at right angles to the zero direction of the ray) through which the spot of light is deflected, and R the horizontal distance of the scale from the mirror in the same divisions, we have $\tan \theta = D/R$, from which θ can be found and hence $\frac{1}{2}\theta$. We have then

$$K \sin \tfrac{1}{2}\theta = (V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right). \quad . \quad (4)$$

where K is a constant.

Equation (1) would be more nearly satisfied if the central portions of the needle to well within the quadrants were as much as possible cut away, leaving only a framework opposite the orifice at the centre of the quadrants to support the needle.

The electrometer, when used heterostatically, admits of a number of different grades of sensibility. These are shown in the two following tables, where L denotes the electrode of the pair of quadrants, one of which is below the induction plate, R the electrode of the other pair of quadrants, I the electrode of the induction-plate, O an electrode of the case of the instrument, and C the electrode of the conductor to be tested. LC denotes that L is connected to C, RO that R is connected to O, RLC that RL and C are connected together, and so on, (L) that the quadrants connected with L are insulated by raising L, (R) that the quadrants connected with R are similarly insulated, (RL) that both L and R are

raised. The disinsulator mentioned above (p. 128) is used to free the quadrants connected with L from the induced charge which they generally receive when L is raised.

GRADES OF SENSITIVENESS.

A.

Inductor connected with quadrant beneath it.

FULL POWER.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} RC \\ LO \end{bmatrix}$$

DIMINISHED POWER.

$$(L)\begin{bmatrix} RC \\ O \end{bmatrix} \quad \text{or} \ (R) \begin{bmatrix} LC \\ O \end{bmatrix}$$

B.

Inductor connected as indicated below.

FULL POWER.
Inductor Insulated.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} RC \\ LO \end{bmatrix}$$

GRADES OF DIMINISHED POWER.

$$(L)\left|\begin{matrix} \begin{bmatrix} RC \\ IO \end{bmatrix} \\ \begin{bmatrix} RIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ RO \end{bmatrix} \end{matrix}\right. \quad \text{or} \ (R)\left|\begin{matrix} \begin{bmatrix} LC \\ IO \end{bmatrix} \\ \begin{bmatrix} LIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ LO \end{bmatrix} \end{matrix}\right.$$

$$(RL)\begin{bmatrix} IC \\ O \end{bmatrix}$$

Either of these grades of sensibility may of course also be varied by increasing the distance of the fibres apart.

The quadrant electrometer can be made to give results in absolute measure by determining the constant C' of equation (3), by which the deflection must be multiplied to give the difference $V_1 - V_2$. This can be done by observing the deflection produced by a battery of electromotive force of convenient amount, determined by direct measurement with an absolute electrometer or by connecting the electrodes of the instrument to two points of a circuit the difference of potential between which is at the same time measured by a standard balance used as a voltmeter (see p. 104). Different such electromotive forces may be employed to give deflection of different amounts and thus give a kind of

calibration of the scale to avoid error from non-fulfilment of condition of proportionality of deflection to difference of potential.

The quadrant electrometer may also be used idiostatically for the measurement of differences of potential of not less than about 30 volts (p. 69). When it is so used the jar is left uncharged, the charging-rod is brought into contact with the inner coating of the jar, and joined by a wire with one of the main electrodes, so as to connect the needle to one pair of quadrants. The other pair of quadrants is either insulated or connected to the case of the instrument. The instrument thus becomes a condenser, one plate of which is movable, and by its change of position alters the electrostatic capacity of the condenser. The two main electrodes are connected with the conductors, the difference of potential between which it is desired to measure.

A lower grade of sensibility can be obtained by connecting the needle through the charging-rod to the electrode R, and using the induction-plate instead of the pair of quadrants connected with L, which are insulated by raising their electrode.

When the instrument is thus used idiostatically V in equation (2) above becomes equal to V_1, and instead of (2) we have

$$D = \frac{c}{2}(V_1 - V_2)^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the deflection is proportional to the square of the difference of potential and therefore independent of the sign of that difference. It is to the left or right according to the electrode connected to the needle. This independence of sign in the deflection renders the

instrument thus used applicable to the determination of difference of potentials in the circuits of alternating dynamo- or magneto-electric generators. (See Chap. IX.)

The quadrant electrometer has been modified by different makers. In a form made in Paris for M. Mascart, the needle is kept at a constant potential by being connected to the positive pole of a dry pile, the negative pole of which is connected to the case, and the replenisher is dispensed with.

In another form devised by Prof. Edelmann of Munich, and suitable for some purposes as a lecture-room instrument, the quadrants are longitudinal segments of a somewhat long vertical cylinder, and the needle consists of two coaxial cylindric bars connected by a cross-frame, and suspended by means of a bifilar. A glass vessel below contains strong sulphuric acid in which dips a vane carried by a platinum wire attached to the needle.

For practical work Sir William Thomson has lately constructed a form of electrometer to be used idiostatically, and has called it an electrostatic voltmeter. It is represented in Fig. 27, and may be described as an air condenser, one plate of which, corresponding to the needle of the quadrant electrometer, is pivoted on a horizontal knife-edge working on the bottoms of rounded V-grooves cut in the supporting pieces. This plate by its motion alters the electrostatic capacity of the condenser. The fixed plate consists of two brass plates in metallic connection, each of the form of a double sector of a circle, which are placed accurately parallel to one another, with the movable plate between them as shown in the figure. The upper end of the movable plate is prolonged by a fine pointer which

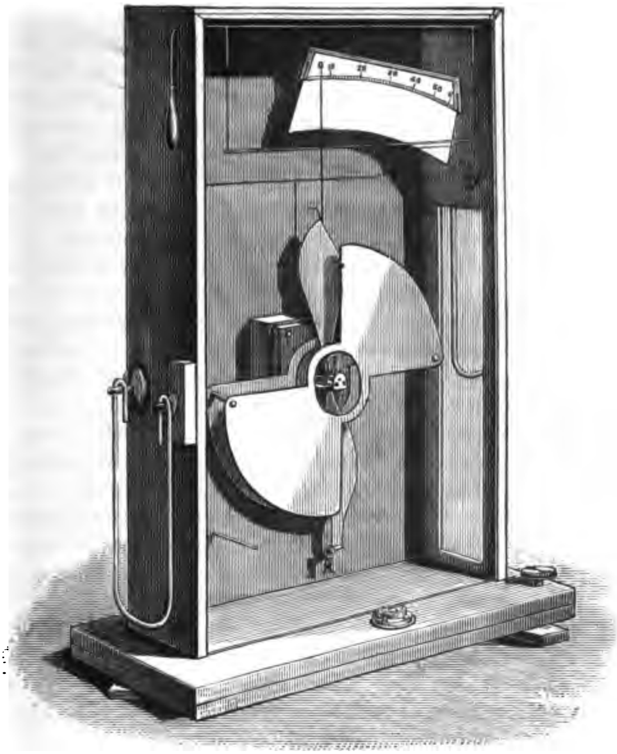

FIG. 27.

moves along a circular scale, the centre of which is in the axis. The fixed plates are insulated from the case of the instrument; the needle is uninsulated.

Contact is made with the plates by insulated terminals fixed outside the case. The two shown on the left-hand side in the figure belong to the fixed plate, and a similar pair on the right-hand side are in connection with the movable plate through the supporting V-groove and knife-edge. The terminals of each pair are connected by a safety arc of fine copper wire contained within a U-shaped glass tube suspended from them and the terminals in front in the diagram which are separated from the plates by the arcs of wire are alone used for connecting to the conductors, or two points of an electric circuit, the difference of potential between which is to be measured, and are therefore called the working terminals.

The instrument is enclosed in a wooden case to diminish the risk of injury to incautious or uninstructed persons who might chance to touch its terminals.

When a difference of potential is established between the fixed and movable plates the latter moves so as to increase the electrostatic capacity of the condenser, and the couple acting on the movable plate in any given position is, as in the quadrant electrometer when used idiostatically, proportional to the square of the difference of potential. This couple is balanced by that due to a small weight hung on the knife-edge at the lower end of the movable plate.

The scale is graduated from 0° to 60° so that the successive divisions represent equal differences of potential. Three different weights, 32·5, 97·5, 390 milligrammes respectively, are sent with the instrument to

provide for three different grades of sensibility. Thus the sensibility with the smallest weight on the knife-edge is a deflection of one division per 50 volts, with the two smaller weights, that is four times the smallest, one division per 100 volts, with all three weights or sixteen times the smallest weight, one division per 200 volts.

The electrostatic voltmeter is graduated as follows. A known difference of potential is obtained by means of a battery of from 50 to 100 cells with a high standard resistance in its circuit. An absolute galvanometer or current balance measures the current in the circuit, and the product of the numerics of the current and the resistance gives that of the difference of potential between the terminals of the latter (see pp. 104, 106). These terminals are connected to the working terminals of the voltmeter, and the deflections noted with the smaller weights on the knife-edge.

For the higher potentials a number of condensers of good insulation are joined in series, and charged by an application of the wires from the terminals of the resistance coil to each condenser in succession from one end of the series to the other. This is done so as to charge each condenser in the series in the same direction, and as the same difference of potential, V say, is produced between the plates of each condenser, the total difference between the extreme plates is nV, if there be n condensers. A convenient large difference of potential can thus be obtained with sufficient accuracy, and being applied to the working terminals of the voltmeter is made to give divisions for a series of different weights hung on the knife-edge. These divisions correspond of course to deflections for known differences of potential with *one* of the weights on the knife-edge.

The divisions thus obtained are then checked by using three instruments which have been dealt with in this way. They are joined in series and a difference of potential established between the extreme terminals, which is observed also by the third joined across the other two. Thus by a process of successive halving and doubling the scale is filled up.

For smaller differences of potential, ranging from 40 to 800 volts, Sir William Thomson has constructed a multicellular voltmeter (Fig. 28) on the same principle. The indicator consists of a number of equal vanes *v*, *v*, ..., in shape similar to the needle of the quadrant electrometer, attached horizontally at equal distances apart on a vertical spindle. The spindle passes at its upper end through a small hole at the centre of a shallow circular box, in the bottom of which is the scale of the instrument. This is covered with a glass plate to guard the indicator from currents of air and keep the scale and other interior parts free from dust.

A vertical brass tube *t*, carries a torsion head *h*, from which the indicator is suspended by a fine platinum wire *w*. Between the upper end of the spindle and the wire is interposed a small coach-spring which having sufficient resilience to allow the spindle to touch a guard-stop, saves the suspension from being broken down by accidental shaking or jolting of the instrument. By turning the torsion head the needle can be adjusted so as to point to zero when no difference of potential is upon its terminals.

Each vane is within a cell similar to that surrounding the vane of the vertical voltmeter but placed in a horizontal position, so that the stationary part of the instrument is a pile of cells or condenser plates *e*, *e*, ... arranged

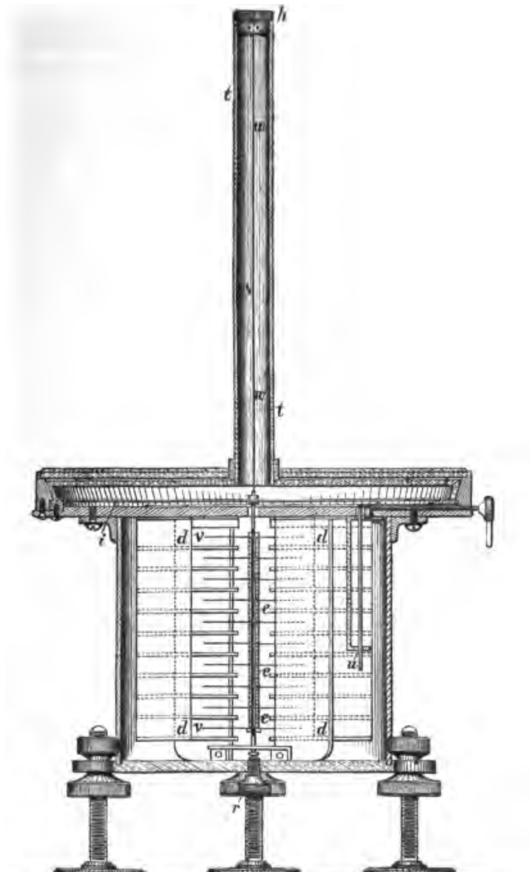

FIG. 28.

vertically above one another. These are all in metallic connection, and between them and the vanes which form the other plate of the condenser the difference of potential to be measured is produced.

Two vertical brass repelling plates *d, d* are attached to the sole-plate of the instrument, and prevent the indicator from turning too far. The lower end of the spindle passes through a hole in a guide-plate carried by these vertical plates, and is prevented from passing back by a little brass head attached to it below. The spindle hangs free by the suspending wire when the instrument is level, and the vanes are then horizontal, each in a plane halfway between the top and bottom of its cell.

The index, *i*, is of aluminium and is attached to the top of the spindle so as to indicate on the scale the difference of potential between the indicator and the fixed plates. The instrument is graduated so as to give volts directly by its readings.

The scale can be read in an engine-room from a distance by means of a mirror placed above the instrument at an angle of 45° with the plane of the scale, giving thus a vertical scale by reflection.

When the instrument is to be carried from one place to another, a thumb-screw *r*, provided below the sole-plate is screwed up so as to take the weight of the spindle off the suspension wire, and clamp the head above referred to against the guide plate.

The instrument is made of four different ranges between the extreme limits of 40 and 800 volts.

The graduation is performed in the following manner. (1) For differences of potential under 140 volts:—A circuit composed of a standard resistance, a standard centiampere balance and a variable resistance joined in

series, is arranged. The multicellular electrometer, properly adjusted, has its terminals joined to the extremities of the standard resistance, and the deflection of its indicator and the current flowing through the balance are both observed. The difference of potential is of course at once got by multiplying the observed current and the standard resistance together. Its value is altered to give different points on the scale by properly changing the variable resistance. Of course the part of the circuit composing the standard resistance and the centiampere balance must be so well insulated as to ensure that all the current which flows through the standard resistance also passes through the centiampere balance.

(2) For differences of potential above 140 volts:—A voltaic pile is used attached by one terminal to one extremity of the standard resistance of the above arrangement, so that the terminals of the multicellular voltmeter attached one to a point in the pile, and the other at the further extremity of the standard resistance, have upon them a difference of potential equal to that between the terminals of the standard resistance plus a constant amount obtained by including the requisite number of couples in the voltaic pile. A previously graduated standard multicellular of the proper grade of sensibility enables this latter difference of potential to be kept constant during the progress of the standardization, while the different points, on the scale of the instrument being standardized, are obtained by varying the difference of potential on the terminals of the standard resistance by means of the variable resistance. Repeating this process the necessary number of times, using as "standard multicellular" first one of the first grade of sensibility (to graduate another of the second grade), then a second

grade, then a first and second grade joined in series, and so on, with greater and greater differences of potential from the pile, we obtain a step-by-step graduation up to the limit of the range of this type of instrument.

Of course when a complete set of standard instruments of this type is available, the ordinary method of graduating by comparison, division for division, may be used; care being always taken to re-test the "standards" from time to time, and to guard them from usage that might tend to produce errors in their measurement of difference of potential.

The pile referred to above is composed of copper-zinc couples, each about an inch square, separated by a layer of blotting-paper moistened with water. The plates are supported on edge, with a diagonal of each plate vertical, on a slotted support, the edges of which are two strips of vulcanite. They are pressed together between two end-pieces of vulcanite by a screw, and are thus kept in position. The papers are wetted by letting the lower edge of the pile dip for a short time into a bath which can be placed below for the purpose.

GRADUATION OF INSTRUMENTS.

We have already considered the graduation of electrostatic voltmeters: we shall now treat, very briefly, the graduation of other instruments for measuring volts and amperes in practical work, and shall take as our examples partly the balances above described and partly other instruments such as ordinary non-absolute galvanometers. The graduation of these instruments may be effected in various ways; for example, by a direct comparison of their indications with those of a standard galvanometer, such as that described

in Chap. IV., or those of an absolute dynamometer, or by such a comparison effected indirectly by keeping a constant current flowing through the instrument and measuring its amount by the electrolytic decomposition which the current produces in a given time in a voltmeter included in the current.

We shall consider first the graduation by the direct comparison method of a potential galvanometer, or galvanometer the resistance of which is so high that the attachment of its terminals to two points in a conductor carrying a current does not perceptibly change the difference of potential formerly existing between these points. Of course every absolute galvanometer measures differences of potential, for, if its resistance is known, the difference of potential between its terminals can be calculated from Ohm's law; but the convenience of a galvanometer especially made with a high resistance coil is that its terminals may be applied at any two points in a working circuit, and the difference of potential, thus calculated as existing between these two points while the terminals are in contact, may, in most cases, be taken as the actual difference of potential which exists between the same points when nothing but the ordinary conductor connects them. For, let V be this actual difference of potential in volts, let r ohms be the equivalent resistance of the whole circuit between the two points and R ohms the resistance of the galvanometer. Then (p. 92) by the application of R, V is diminished in the ratio of R to $R + r$, and therefore the difference of potential between the ends of the coil is now $VR/(R + r)$. Hence by Ohm's law we have for the current through the galvanometer the value $V/R(1 + r/R)$. If r be only a small fraction of R, r/R is inappreciable, and the difference of potential

calculated from the equation $C=V/R$ will be nearly enough the true value. So far, it is to be observed, r is the equivalent resistance between the two points, and the result stated holds, however the electromotive force may have its seat in the circuit, if only R be great in comparison with r. If, however, either of the two parts of the circuit between the two points in question have a resistance r, then as shown at p. 94 above, if only R be great in comparison with r the value of the difference of potential between the terminals of r is practically unchanged by the addition of R as a derived circuit.

The instrument to be graduated is first tested as to the adjustment of its coil, needle, &c. We shall suppose in the first place that the current through its coil is proportional to the tangent of the deflection, according to the conditions stated above (Chap. IV.), as necessary for a tangent galvanometer. The standard galvanometer and it are then properly set up with their needles pointing to zero, and their coils in the magnetic meridian, in positions near which there is no iron, and at which the values of H have been determined. The high resistance coil of the standard galvanometer and the coil of the potential instrument are joined in series with a constant battery of as many Daniell's cells as gives a deflection of about 45° on the standard galvanometer, or a conveniently readable deflection if a mirror and scale are used, and a deflection of the needle also of nearly 45° on the potential instrument. If, however, the instrument, as often happens, is very much more sensitive than the standard, the method next described (p. 149) may be employed. The current actually flowing in the circuit is calculated by equation (11) or (12) from the reading obtained on the standard, and reduced to amperes by

multiplying the result by 10. Care of course must be taken that there is no leakage in the circuit which might cause the current flowing through the standard galvanometer to be different from that flowing through the other instrument. The difference of potential between the two ends of the coil of the potential galvanometer is found in volts by multiplying the number of amperes thus found by the resistance of the coil in ohms. From this is found the number of divisions of deflection which corresponds to one volt between the terminals when the needle is in any other field of known intensity. We can then obtain, by an obvious calculation, the number of divisions of deflection which corresponds to one volt between the two ends of the coil, and thence from the value of H the number of divisions which would correspond to one volt if the intensity of the field were one C.G.S. unit. For example, let the deflection be 40 divisions for 20 volts, and let the value of H at the instrument which is being graduated be ·17. The number of divisions of deflection which would correspond to one volt for that position, if the field were of unit intensity, would be $\frac{40}{20} \times {\cdot}17 = {\cdot}34$. Hence if the instrument be placed in a field where the horizontal intensity is ·185 the number of divisions of deflection which would correspond to one volt would be ·34/ ·185 = 1·838.

Another method sometimes convenient is as follows. The standard instrument, a few good Daniell's cells, and a resistance which makes the deflection about 45° on the standard, are joined in series, and the galvanometer to be graduated is applied at two points in the circuit which include between them such a portion of the resistance as gives a deflection of about the same amount. Let R ohms be the portion of the resistance included between

the terminals of the galvanometer, and let G ohms be the resistance of the galvanometer coil. Let the current calculated from the deflection on the standard be C amperes, then if V be the difference of potential in volts between the terminals of the potential instrument, we have by Ohm's law—

$$C = \frac{V}{R'},$$

where R' is the resistance equivalent to the divided circuit of R and G. But (p. 87) $R' = RG/(R + G)$, and therefore

$$C = V\frac{R + G}{RG}.$$

Hence,

$$V = C\frac{RG}{R + G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This last equation gives the number of volts indicated by the deflection on the potential instrument, for the field in which its needle is placed; and from this, in precisely the same manner as described above, the number of volts for any other field or position of the needle may be found.

If the scale of the instrument does not follow the tangent law, it is necessary to determine by direct experiment the electromotive force corresponding to different deflections and thus, so to speak, calibrate the instrument. To do this the most convenient plan is to divide the scale accurately into equal divisions and to number these from zero at the position of equilibrium with no current. Then the current measured by the standard galvanometer is varied conveniently by introducing resistance into the

current by a rheostat, and the deflection observed for several different values. The corresponding differences of potential are then plotted on squared paper as ordinates for which the numbers of divisions of the deflections are the corresponding abscissæ. A curve is then carefully drawn through the extremities of these ordinates and the ordinate of this curve drawn for any chosen abscissa will be the difference of potential for that deflection.

The reason for using a deflection of 45° in these experiments is this, that a given absolute error in reading the deflection gives at that deflection a minimum percentage error in the estimation of the current.

For verifying the accuracy of the graduation of the potential instruments when performed by either of these methods, a standard Daniell's cell of the form proposed by Sir William Thomson at the Southampton meeting of the British Association, or a standard Clark's cell of the form, and prepared with the precautions, recommended by Lord Rayleigh (see p. 153 below), may be used.

The Daniell's cell is represented in the annexed cut (Fig. 29). It consists of a zinc plate at the bottom of the vessel resting in a stratum of saturated zinc sulphate, on which has been poured, so gently as to give a clear surface of separation, a stratum of half saturated sulphate of copper solution, in which is immersed a horizontal plate of copper. The copper-sulphate solution is introduced by means of the glass tube shown in the diagram dipping down into the liquid, and terminating in a fine point, which is bent into a horizontal direction so as to deliver the liquid with as little disturbance as possible. This tube is connected by a piece of india-rubber tubing with a funnel, by the raising or lowering of which the sulphate of copper can be run into or run out of the cell. By this means the

sulphate of copper is run in when the cell is to be used, and at once removed when the cell is no longer wanted.

Daniell's cells show considerable variation of electromotive force with different solutions, and therefore the liquids should not be twice used. The solutions should be carefully prepared, and kept in stock bottles till wanted. In all cases careful determinations of the

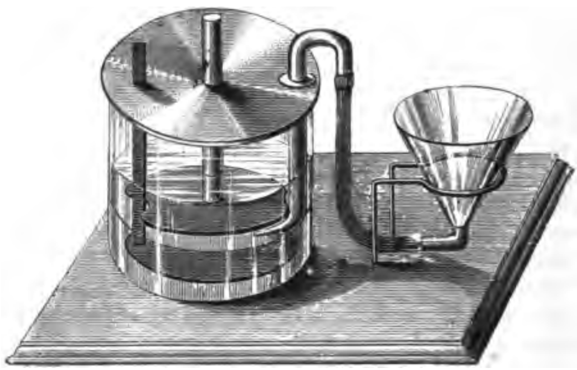

FIG. 29.

electromotive force should be made from time to time. The electromotive force of this cell has been determined very carefully and found, according to Lord Rayleigh's latest determination of the ohm, to be 1·072 volt at ordinary temperatures. The temperature at which it has been employed has usually been about 12° C. The cell is so used (p. 155) that a feeble current flows through it, and a feeble current should be passed for a little time just before

to freshly coat the copper plate. The variation of electromotive force with temperature has not been determined, but has been found not to cause any perceptible error for the small changes of temperature to which the cell was exposed. The direct application of this cell to the galvanometer gives at once a check on the graduation. If the resistance of the galvanometer is always over, say, 6,000 ohms, there is practically no polarization.

The behaviour of Clark's standard cell has been studied with great care by Lord Rayleigh. It may be made in a reliable and handy form in the following way, which includes the precautions that Lord Rayleigh's experience has shown to be necessary. The vessel is a weighing tube, or for small sizes merely a test-tube, with a platinum wire sealed through the bottom, and rests on a suitable stand as shown (Fig. 30). This wire makes contact with mercury, which occupies the bottom of the cell and forms one of the plates. The mercury must be pure, and it is desirable to ensure its being so by redistilling in vacuo good clean commercial mercury. On the mercury rests a paste made by adding to 150 grammes of mercurous sulphate 5 grammes of zinc carbonate, and sufficient saturated zinc sulphate solution to give a stiff pasty consistency.

The zinc sulphate solution should be made from pure zinc sulphate dissolved under gentle heat in distilled water so as to make a saturated solution, and, after having been allowed to stand for some time to precipitate any iron which may have been present in the sulphate, filtered in a warm place into a stock bottle. When required the solution is gently warmed, and drawn off by a siphon from just above the crystals at the bottom. The paste is made by placing the mercurous sulphate and zinc carbonate in a mortar and rubbing in the zinc sulphate at

intervals during two or three days, to give time for all carbonic acid to pass off.

A rod of what is called "redistilled zinc" resting in the paste, and held upright in the vessel by a notched ring of

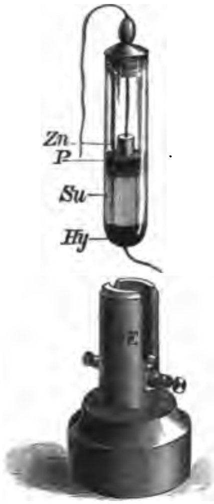

FIG. 30.

cork, forms the other plate. The zinc is cleaned before putting it in position by dipping it in sulphuric acid and then washing it in distilled water. Connection with it is made by a gutta-percha-covered copper wire soldered to it, and passed up through a cork which closes the cell and nearly fills the upper part of it so that very little air is included. The cork is flush with the top of the tube,

and the edges of the tube and the whole upper surface of the cork is covered with marine glue to seal up the cell.

A cell thus made, if used with only the very feeblest currents, never short-circuited, nor exposed to great variations of temperature, will have a constant electromotive force E in true volts at temperature t° C., given according to Lord Rayleigh's determination by the equation

$$E = 1{\cdot}435\left\{1 - {\cdot}00077\,(t - 15)\right\}.$$

The standard Daniell's cell is very conveniently used along with a Daniell's battery in the manner represented in the diagram, Fig. 31. C is the standard cell, and B a battery of from 30 to 40 small gravity Daniells.* A circuit is formed of a resistance box, the galvanometer G to be graduated, and the battery B joined in series with the standard cell C. A sensitive galvanometer D, which may be a reflecting galvanometer, or any very sensitive galvanometer of low resistance, has one terminal attached at a point M between the battery and the standard cell, and the other terminal through the key K to an intermediate terminal L of the resistance box. The resistances in the box, on the two sides of L, are adjusted until no current flows through the galvanometer D, when the key K is depressed.

Let R be the resistance in the box to the right of L, r

* These can be very easily made by using large preserve-pots as containing vessels, and placing at the bottom of each a copper disc of from three to three and a half inches in diameter, in a stratum of saturated copper sulphate solution, and a grating or plate of zinc a little below the mouth of the vessel immersed in a solution of zinc sulphate, of density 1·2. The copper sulphate may be kept saturated by crystals dropped into a glass tube passing down through a hole in the zinc plate to the copper. A copper wire well covered with gutta percha should be used as the electrode of the copper plate.

the resistance of the cell C, and G the resistance of the galvanometer. Then if V be the difference of potential, in volts, between the terminals of the galvanometer,

$$V = 1{\cdot}072 \frac{G}{R + r}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In practice a resistance of from 300 to 400 ohms is generally required for R. The electromotive force of the standard cell is taken as 1·072, as, notwithstanding the large

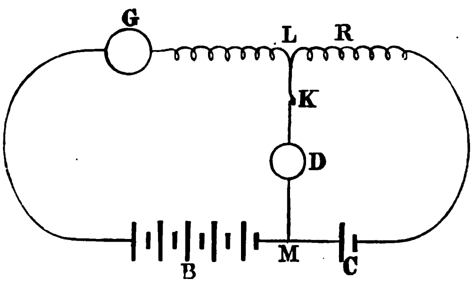

FIG. 31.

battery in the circuit, the total resistance is so great that there is very little polarization. This method in fact is peculiarly well adapted for the Daniell's cell, as the slight current flowing through serves to keep its plates in a constant and fresh state.

The difference of potential, the magnitude of which is thus obtained, is chosen such as to give a convenient deflection on the instrument to be graduated.

Another method of using the standard cell for determining the difference of potential given at the terminals of the galvanometer G by the battery B, and one better adapted for a standard like Clark's through which it is not desir-

able to send even a slight current for any length of time, is obtained by placing R between L and M in Fig. 31, K and D in the position occupied by R, and reversing the cell C. R is adjusted until no current flows through D when the key K is tapped down for an instant. When this is the case the electromotive force of C is balanced by the difference of potential at the two ends of R produced by B. Hence the difference of potential in volts then existing between the terminals of G is given (for a Clark's cell at 15° C.) by the equation,

$$V = 1{\cdot}435 \frac{G}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By this method, which is an application of Poggendorff's method of comparing the electromotive forces of batteries, balance is obtained when no current is flowing through the standard cell, and disturbance from polarization is altogether avoided. It has been found very easy and convenient in practice.

The methods first described (pp. 147—151) are of course applicable to the graduation of current galvanometers or balances. The standard galvanometer, of which in this case the low resistance coil is used, and the current galvanometer to be graduated are joined in series with a battery, which with some resistance in circuit is sufficient to give a deflection in each of about 45°, if the indicators are pointers, or as large as possible a deflection in each case as can be conveniently obtained within the ranges of the instruments if their utmost range is less than 45°. The current flowing in amperes is given by the standard, and this, of course, is the number of amperes which is indicated by the deflection of the current instrument. By an obvious calculation from the value of H at the current

instrument, precisely similar to that above described, the number of divisions of deflection corresponding to a current of one ampere for a field of unit strength is found, and from this the deflection corresponding to one ampere with any given field.

In some instruments the field at the needle of the galvanometer is varied in order to obtain greater or less sensibility by superimposing on the earth's field a horizontal field due to a permanent magnet of hard steel (as in the magnetostatic current-meter described at p. 112 above), or to an electromagnet. The value of the field intensity given by this magnet at the needle of the galvanometer when in position, may be determined in the following manner. A battery of about 30 of Sir William Thomson's Tray Daniells is joined in series with a resistance, R, of about 7,000 ohms. The electrodes of a potential galvanometer,* in which the needle is acted on only by the earth's field, H, are attached at two such points in this resistance that the deflection of the needle produced is from 30 to 40 divisions on the scale. The current through the galvanometer is now stopped and the magnet placed in position, so that the index of the galvanometer is again at zero. The electrodes are now placed so as to include a resistance which makes the deflection nearly what it was in the former case. Let E be the electromotive force, and B the resistance of the battery, I the total horizontal intensity of the magnetic field at the needle when the magnet is in position; R_1, R_2 the resistance included between the electrodes of the galvanometer in

* This, according to convenience, may be the instrument being tested, or another which allows the magnet to be placed in the same position relatively to the needle.

This and the other methods described below are applicable to the comparison of the sensibilities of the magnetostatic current-meter for any two positions of its field magnets.

the first and second cases respectively; V_1 and V_2 the potential difference in volts on the instrument in the same two cases; D_1 and D_2 the corresponding deflections, and G the resistance of the galvanometer. By Ohm's law

$$V_1 = \frac{E\,R_1\,G}{(B+R-R_1)(R_1+G)+R_1\,G} = m\,H\,D_1,$$

$$V_2 = \frac{E\,R_2\,G}{(B+R-R_2)(R_2+G)+R_2\,G} = m\,I\,D_2,$$

where m is a constant. Therefore we have

$$I = H\frac{D_1\,R_2\,\{(B+R-R_1)(R_1+G)+R_1\,G\}}{D_2\,R_1\,\{(B+R-R_2)(R_2+G)+R_2\,G\}}. \qquad (8)$$

If the resistance B of the battery be small in comparison with G, or if the galvanometer be sensitive enough to allow B/G to be made sufficiently small by resistance added to G, B may be neglected; and it is generally possible, by properly choosing R, R_1 R_2, to simplify very much this formula. The number I thus found, diminished by H, is the number of C.G.S. units which measures the horizontal intensity of the magnetic field produced at the needle by the action of the magnet alone. The value of $I-H$ obtained should be carefully verified from time to time, and the value after such determination marked on the magnet.

The known difference of potential obtained by comparing a Daniell's battery with the standard cell by the methods described on pp. 155—157, may be used to give a deflection with the magnet in position, and this deflection compared with that obtained at the same position of the galvanometer, with the earth's force alone, and the standard cell directly applied to the instrument. From this the ratio of I to H can be at once obtained.

The value of I may be found at any place and for any magnet by means of a potential galvanometer, as follows.* Set up the instrument and apply a difference of potential of V volts (measured by comparison with a standard cell as described in p. 156) to the terminals of the coil. Let the deflection produced be D scale-divisions, and let n be the number of divisions (determined as already explained) which would be given by one volt if the field were of unit strength; then, $I=n\ V/D$.

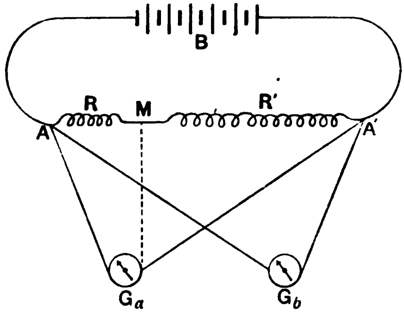

FIG. 32.

Denoting the electromotive force of the standard cell in volts by E, we have as in p. 157, $V=EG/R$, and therefore $I=n\ EG/DR$.

The intensity of an artificial magnetic field at the needle of a galvanometer may also be determined as follows. A circuit is arranged containing as shown in Fig. 32 a battery of about 30 Daniell's elements and

* This method is very convenient for testing the constancy of the field magnets from time to time. Such tests must of course be made with a reliable standard cell, and at a place remote from magnets and iron.

two resistances R, R' of about 100 ohms and 600 ohms respectively. Two potential galvanometers, G_a, G_b, are connected to the points A, A' of the circuit; of these G_a has the artificial field at the needle which it is proposed to measure, the other has a magnetic field of convenient intensity, and we shall suppose that with the arrangement made a deflection of from 30 to 40 divisions of the scale is obtained on each instrument. Calling D_a the deflection of the galvanometer G_a, and D_b that of G_b, then as the two instruments measure the same difference of potential V,

$$V = AID_a = BD_b \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which I is the field intensity at the needle of G_a, and A, B are constants.

The artificial field is removed from G_a, the index of G_a brought to zero, and its terminals placed on the smaller resistance R, while G_b is left as it was. The deflections now produced, D'_a, D'_b, are read off. If now V' is the difference of potential between the terminals of G_b, that between the terminals of G_a has the value

$$\frac{V' \dfrac{RG}{R+G}}{R' + \dfrac{RG}{R+G}} = \frac{V'}{1 + \dfrac{R'}{R}\left(1 + \dfrac{R}{G}\right)} = AHD'_a . \quad (10)$$

where G represents the resistance of the galvanometer G_a. This gives us the relation

$$V' = AHD'_a \left\{ 1 + \frac{R'}{R}\left(1 + \frac{R}{G}\right) \right\} = B'D'_b \quad (11)$$

Equations (9) and (11) give

$$I = H\frac{D'_a D_0}{D_a D'_b}\left\{1 + \frac{R'}{R}\left(1 + \frac{R}{G}\right)\right\} \quad . \quad (12)$$

I—H gives in C.G.S. units the intensity of the artificial magnetic field at the needle.

When the instrument has been graduated for a field of intensity equal to 1 C.G.S. unit, and the intensity of the field given by the magnet at the needle has been determined, the graduation is complete. In the practical use of the instrument with the magnet in position, the number of volts, or the number of amperes (according as the instrument used is a potential or a current galvanometer) corresponding to a deflection of any number of divisions is found by the following rule :—

Multiply the number of divisions in the deflection by the intensity of the magnet's field increased by the horizontal intensity of the earth's field, and divide by the number of divisions in the deflection which corresponds to one volt in a field of one C.G.S. unit intensity.*

When the magnet is not used the rule is the same as the above, except that the divisor to be used is the value of H for the place of the galvanometer.

The graduation of the current galvanometer may also be performed by means of electrolysis. The electro-chemical equivalents of a large number of the metals have been determined, and it is only necessary therefore, in order to graduate the instrument, to join it in series with a proper electrolytic cell and a constant battery, and to compare the amount of metal deposited on the negative plate of the cell with the total quantity of electricity which flows through the circuit in a certain time.

* The mean value of H for Great Britain may, *when the magnet is used*, be taken with sufficient accuracy as ·17 C.G.S.

The two best electrolytes for this purpose are solutions, of proper strength, of nitrate of silver and of sulphate of copper. With the former solution electrodes of silver, with the latter solution electrodes of copper are used. When proper precautions are taken very reliable and accurate results can be obtained with either electrolyte. These precautions relate to the preparation of the solutions, the adaptation of the size of the plates to the strength of the current passing through the voltameter, and the washing and drying of the plates before being weighed.*

A form of cell very convenient for use with both solutions when the current strength is not greater than 10 amperes, is shown in Fig. 33. It consists of three parallel plates of pure silver, or pure copper, suspended from spring clips in a glass vessel containing the proper solution. This form of cell has the advantages of giving light plates, which facilitate the accurate weighing of the amount of loss or gain of metal, and of allowing, when silver is used, and the size of the plates is properly proportioned, the loss from the anode to be used as a check in estimating the gain on the kathode. There is of course the objection which attends the use of vertical plates, that the solution becomes less dense near the kathode, but the only practical effect due to this has been found to be a slightly greater thickness of deposit in the lower part of the plate due to the greater density there.

Lord Rayleigh has used as voltameter a platinum bowl as kathode, and a silver plate as anode. This cell, though it has several advantages, is more difficult to

* See a paper on the 'Electrolysis of Silver and Copper,' T. Gray, *Phil. Mag.* Oct. 1886, from which the details here given are mostly taken. See also a paper by A. W. Meikle, *Electrical Engineer*, Mar. 23, 1888.

manipulate than that described above. The anode must be protected to prevent silver from falling from it to the kathode below. The amount of silver deposited may be determined in two ways, both of which should be used so that one may be a check on the other: (1) the platinum bowl is previously carefully weighed, and again with the deposit adhering, (2) the silver is dissolved from the

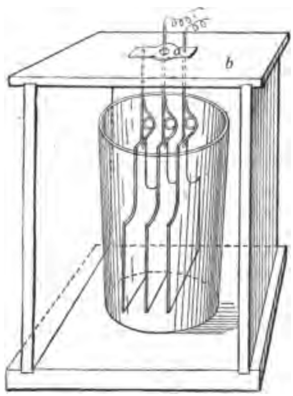

FIG. 33.

platinum bowl by nitric acid, and its amount afterwards estimated.

The form of clip, as illustrated in Fig. 34, almost explains itself. It is made of stiff platinoid or brass wire. A piece is taken of the proper length, bent into a close loop at the middle, then each half wound two or three times round a rod of metal to form springs as shown, and the two ends bent round to meet side by side,

and there soldered to a stiff back-piece of brass. The springs when soldered in position should cause the loop to press firmly against the back-piece so as to form a firm clip.

The stems of the two outer clips when in position are connected by a cross-piece *a* of copper. Both are insulated from the inner clip by a block of vulcanite through which its stem passes. This whole arrangement

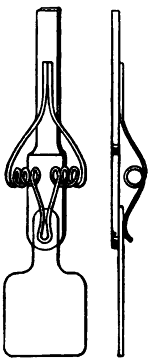

FIG. 34.

of cross-piece and insulating block is fixed on the top *b* of the wooden framing shown in Fig. 33.

The two plates attached to the outer clips form the anode of the electrolytic cell, and the plate between them the kathode. The kathode thus gains on both sides, and as it is safer to use the gain than the loss of metal in estimating the current, the weight of the plate itself is thus made as small as possible in comparison with the alteration in weight to be determined.

Since the form of cell shown in Fig. 33 was arrived at it has been improved by the substitution for the cover *b* of a rectangle of wood, well soaked in paraffin or varnished, which carries on one side the clips for the anode, and at the middle of the opposite side the single clip for the kathode.

When currents of over 10 amperes are to be used the form of cell shown in Figs. 35, 36 is preferable. An insulating rim rests on the top of the cell, which for the larger

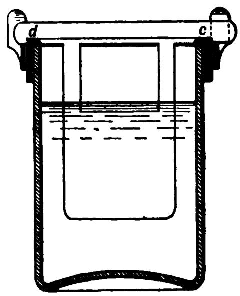

FIG. 35.

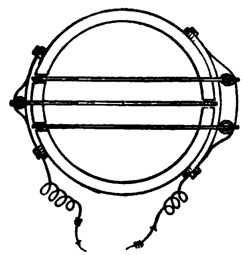

FIG. 36.

sizes is conveniently made of earthenware and of rectangular shape. A groove in the rim fits the top of the cell loosely so that the rim with its attachments can be easily removed and cleaned. To the rim are fixed on opposite sides two sets of spring clips, each made as shown in Fig. 37, by soldering flat strips of springy metal to a stiff base-piece which can be screwed to the insulating rim of the cell. To make the effective area of the plates as great as possible in comparison with the ineffective part, the part above the liquid is cut away to

two narrow strips connecting the lower part to an upper cross bar *c*, *d*. One end *c* of this cross-bar rests in a clip, the other in a notch in the insulating rim. Anode plates and kathode plates alternate with one another, and there is one more of anodes than of kathodes, so that each kathode is between two anodes. In large cells where the plates are close and liable to touch, they are kept apart by two *U*-shaped glass tubes hung over each alternate plate.

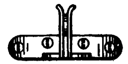

FIG. 37.

With regard to the size and preparation of plates experience has shown that in the cases of both silver and copper there is a certain density of current (current strength per unit of area of plate) which gives the most adherent and in the case of silver most finely crystalline deposit. When silver is used there is a tendency, if the plate be too large or two small, for the crystals of deposited silver to grow out branch-like from one plate to the other, an effect which is most marked where there is a sharp edge or corner. Hence the plates must have their edges and corners rounded off to prevent the formation of these "trees," which cause great risk of loss of silver from the plate in its treatment before being weighed.

The best deposit has been found to be obtained with a solution made with five parts by weight of nitrate of silver to 95 of water, and a kathode plate giving not more than 600 sq. cms. nor less than 200 sq. cms. of active face to the ampere of current. If a stronger solution be used, the density of current may be somewhat increased, but the strength should not be less than 4 per cent. nor greater than 10 per cent.

The anode plates should be considerably greater in

area than the kathode plates if their surface is to remain bright and moderately hard so as to admit of the plates being weighed if necessary. The density of the current for them should be less than one ampere to 400 sq. cms.

If the anodes are of rolled sheet silver the surface skin should be polished off with fine silver sand, and the plate washed in distilled water before being used; as otherwise the silver would be dissolved away from under the skin, which would hang as a loose sheet ready to break away when the plate was moved. A plate of silver becomes soft and inelastic by repeated use as an anode, owing to solvent action going on below the surface, and to remedy this should be heated after being used each time to a red heat in the flame of a spirit lamp.

The following mode of treating silver plates has been found very successful. The plate cut from the new sheet has its corners first rounded and smoothed, then is polished with fine silver sand in water, rubbed on with a soft clean pad of cloth, so as to remove the skin above referred to, and still leave a smooth surface. A gentle stream of clean water is then run over the surface from a tap to remove the sand, next the plate is washed, first with clean soap and water, then with water alone, then immersed for a few minutes in a boiling solution of cyanide of potassium, and finally washed thoroughly in a stream of clean water. The plate is dried in a current of hot air, for example before a clear fire; and great care must be taken in handling it after it has been cleaned not to touch it with the fingers, otherwise the parts which have been in contact with the skin will receive no deposit. Of course the plate must be allowed to cool before it is weighed to obviate risk of disturbance from air currents in the balance case.

When the silver deposit is to be washed and weighed, the plates are gently removed by easing the springs to prevent risk of rubbing off metal by the friction of the clips, then dipped gently in clean, recently distilled water contained in a glass vessel, so that any small crystals which may fall from the plate may be detected. The adherent nitrate solution is thus to a great extent removed; and the plates are then laid in the bottom of a shallow glass tray containing clean distilled water, and washed by gently tilting one side then the other of the tray so as to make the water flow gently over their surfaces. Then they are washed in a second tray in the same way, and allowed to soak for a quarter of an hour before being dried.

To dry the plates one corner is laid on a pad of blotting-paper and the greater part of the water drained off. The plate is then dried by holding the upper end in a spirit flame.

The electrolysis of copper sulphate with copper anode and kathode gives results which for very high accuracy in standardizing are but little if any inferior to those obtained with silver: for most practical purposes results quite accurate enough can be obtained with much less experimental skill on the part of the operator. Repeated experiments made in the Physical Laboratory of the University of Glasgow, in connection with the graduation of Sir W. Thomson's standard instruments,* have shown that under certain easily fulfilled conditions the method of standardizing by the electrolysis of copper sulphate is accurate and trustworthy.

* See the Ref. p. 163 above. The remarkable concordance of standardizings made at different times is illustrated by results quoted in Mr. Meikle's paper.

The size of plates is not of so great importance as in the case of silver, but the kathode plate for the best results in long-continued electrolyses should have about 50 cms. of active surface or upwards per ampere. When the current is of small density deposits are obtained which are much more solid and adherent than those of silver, and therefore much more easily dealt with. As in the case of silver the anode should be of much greater area than the opposed surface of the kathode. With a density of current of upwards of $\frac{1}{30}$ of an ampere per sq. cm. the resistance at the anode becomes variable and very considerable, sometimes almost stopping the current, which after a little, with evolution of gas at the anode, regains nearly its former strength.

It was found by Prof. T. Gray in the experiments above referred to that satisfactory and concordant results could be obtained with a solution of any ordinary pure commercial copper sulphate made with pure water, provided the density did not fall below 1·05, and the solutions were made slightly more acid than in their normal state. An addition for example of $\frac{1}{10}$ per cent. of sulphuric acid to different solutions, which gave results differing among themselves, brought them into complete accordance. The loss of weight which is well known to take place when a copper plate is left standing in a copper sulphate solution, was also carefully investigated. This loss it was found seldom exceeds $\frac{1}{200}$ of a milligramme per sq. cm. per hour, or about $\frac{1}{5000}$ of that which would be deposited by a current of one ampere per 50 sq. cms. When the current density is smaller than this the loss is nearly the same as when no current flows. The effect seemed to have a minimum for a density of solution between 1·10 and 1·15, and seemed for

this density to be rather retarded than the reverse by the addition of a small percentage of free acid.

The kathode plate having been cut and rounded at the corners is polished with silver sand in the same manner as the silver plate. It is then placed in the cell and a thin coating of copper deposited over it, while the current (if a large current is to be used), is adjusted to its proper strength by placing resistance in the circuit. The plate is then removed, washed in clean water and dried before a clear fire without being sensibly heated. Any defect in the first cleaning will be shown by the deposit, and if no such defect is shown, the plate is weighed and replaced in the cell for the continuation of the electrolysis. If feeble currents are to be used this preliminary adjustment is hardly necessary, as it is preferable then to use a larger number of cells than are absolutely necessary to produce the current, and bring down the current to the necessary strength by adding an amount of resistance which can be easily enough estimated.

After the electrolysis the plates are carefully removed and at once dipped in ordinary (not necessarily distilled) clean water, containing two or three drops of sulphuric acid per litre, then washed in a tray like the silver plates. The plates are then rinsed in clean water without acid, and dried first in a clean pad of white blotting paper, and then before a fire or over a spirit lamp. If this is carefully done and the deposit be fairly good no copper will be lost and there will be no gain of weight by oxidation. The plates may be weighed after having been allowed to cool down to the ordinary temperature.

The anode plates are treated in a similar manner (except as regards the drying in a blotting-pad, which might cause loss of silver) without loss of copper, or gain

by oxidation, but owing to loss of weight in the solution, &c., they give much less satisfactory results than do the kathode plates.

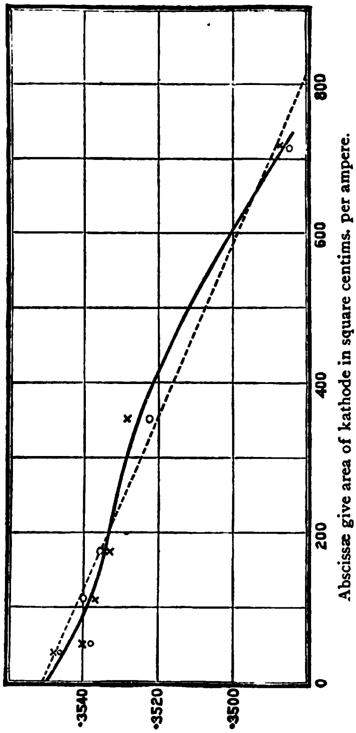

FIG. 38.

The arrangement of the circuit for electrolytic experiments consists of a battery of tray Daniells, or other

constant cells, joined in series with the electrolytic cells to be used, a sensitive galvanometer, and a rheostat (or other readily variable resistance) by which the current is to be regulated. The current is adjusted so that a convenient deflection is obtained, which is restored by slightly turning the rheostat in the proper direction, if any alteration takes place. The conduct of an experiment will be understood from the description of the process of standardizing given below.

Lord Rayleigh has carried out the electrolysis of silver nitrate with very great care, and measured the electro-chemical equivalent of the metal, that is the amount of silver deposited per absolute unit of electricity passed through the voltameter. He found that a coulomb deposits very approximately ·0011179 gramme of silver.*

From this result Professor T. Gray has experimentally determined by comparison the electro-chemical equivalent of copper, and found it to be very approximately ·0003287 (or for practical purposes ·0003290) at ordinary temperatures, and with a current density of one ampere per 50 sq. cms. of active surface of kathode. This number can be corrected for other current densities by the dotted curve given in Fig. 38.

The results from which this curve has been plotted are given in the Table on the following page.

The effect of variation of temperature † on the amount of copper deposited has been found by Mr. A. W. Meikle to be very slight at ordinary temperatures; for a change from 12° C. to 28° C. it is a diminution for a given size of plate of only $\frac{1}{10}$ per cent.

At temperatures rising above 20° C. the effect of variation of size of plate becomes more and more important.

* *Phil. Trans.* Pt. II. 1884. † See Ref. p. 163 above.

AMOUNTS OF COPPER DEPOSITED BY THE SAME QUANTITY OF ELECTRICITY ON KATHODE PLATES OF DIFFERENT AREAS.

Area of plate in sq. cms.	Amount of deposit in grammes (first experiment).	Amount of deposit in grammes (second experiment).
3	·3534	·3533
5	·3530	·3529
11	·3528	·3530
18·5	·3526	·3527
36	·3524	·3521
73	·3503	·3502

The application of electrolysis to the standardizing of instruments will now be illustrated by a short account of its application to the determination of the proper weights for use in the current balances described above. The arrangement of apparatus is shown in Fig. 39 which may be taken as a plan of the standardizing table with instruments in position. C C C C C C are six of the Electric Power Storage Co's secondary cells, shown joined in series, by being connected to a series of mercury cups *m*, *m*, . . . which are connected across by thick copper rods as indicated by the full and dotted lines. (These cups are on a vulcanite base, and have bottoms of thick copper to ensure contact.) When however currents of great strength are required for the graduation of low resistance instruments, these cups are joined in parallel by two rods of copper which have teeth at the proper distance apart to fit into the cups, so as to join all in each row together. The battery fully charged and thus joined in parallel will maintain a current of 200 amperes for 10 hours.

The terminal cups of the commutating board are shown joined to a distributing board provided with cups, 1, 2, . . . 12, by which the battery is put in series with a rheostat R, in parallel arc with a set of conductance bars *D*, a galvanometer *G*, a pair of large electrolytic cells

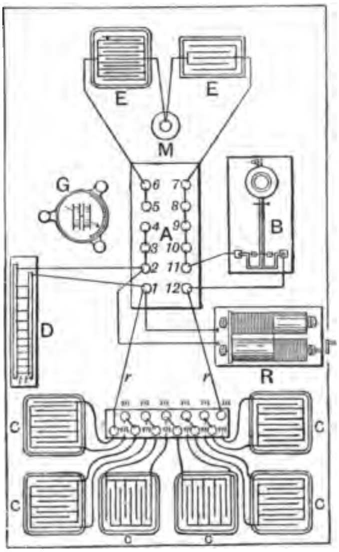

FIG. 39.

joined by a movable cup *M*, and finally the balance *B* to be standardized. The conductance bars are constructed as shown in Fig. 40. Rods of platinoid of thickness according to the conductance required are bent into *U*-shape as shown, and the limbs held at proper distances

apart by wooden blocks at intervals, or by a strip of wood running along their whole length, according as the rods are thick or thin. The length of rod in each *U* is about 4 metres, and the thickness is chosen such that one or two volts difference of potential produces very little heating of the wire. The troughs, *tt* (Fig. 39), are made with bottoms of thick copper and contain mercury in which

FIG. 40.

the ends of the rods (or thick copper pieces soldered to the wires if thin) rest pressed down by their own weight. The different *U*s beginning from one side are graduated so as to have conductances nearly in the ratios 1 : 1 : 2 : 4, &c. so that the total conductance in the set may be increased at will by a step equal to the lowest conductance (since each conductance is that amount greater than the sum of all that precede it in the series). When

any bar is not in use its lower ends are lifted out of the troughs as shown in the Figure. The rheostat, which has a least conductance rather less than that of the smallest bar, furnishes an auxiliary variable bar by which the conductance can be gradually altered. Its wire is of stranded copper and can carry 10 amperes without damage.

The current balance has previously had its scale graduated and attached as described above, and it remains only to show how the constant of the instrument is determined, or in other words the weight which placed on the beam will enable the current to be obtained from its indications in the manner already described (p. 102). A chosen arbitrary counterpoise weight is placed in the trough, and another, which then just brings the beam to the sighted position without current when at the zero of the scale, is placed on the beam with the index at some division near the right-hand end so that a current of, say, 10 amperes (more or less according to the instrument) is required to bring the beam to the sighted position. The electrolytic cells are then arranged to give about 500 sq. cms. of kathode surface, and are joined up with a conductance sufficient to give nearly the required current. The balance will come nearly to zero, and is brought to zero exactly by adjusting the current by means of the rheostat. These adjustments having been made, the kathode plates are removed, washed, weighed, and replaced. At an instant observed on an accurate timekeeper the circuit is closed, and any deviation of the current corrected by means of the rheostat. The current is brought to its correct value in from five to ten seconds, and hence in an electrolysis of say an hour (the usual duration of an experiment) the error due to its deviation

from the final constant value for this short variable period is quite imperceptible. Any variations of the current strength are observed on the instrument itself, or if (which rarely happens) that is not sensitive enough, on a more sensitive galvanometer *G* (Fig. 39), which is introduced when required, and kept out of circuit at other times. Any sufficiently sensitive instrument which can have its (not necessarily known) constant changed by any required amount by varying the field at the needle, or by using an instrument provided with two parallel coils with the needle midway between them, and arranged to permit the distance of the coils apart to be altered at pleasure, is convenient for this purpose.

The electrolysis having thus been carried on and completed, the circuit is broken, and the plates washed and weighed. The current is calculated from the result by dividing the gain of copper on the kathode expressed in grammes, by the electro-chemical equivalent of copper (·0003287, or, as explained above, the proper value for the density of current), and the result by the number of seconds during which the electrolysis has lasted. Let C be the current for the position of the weight on the beam as given by the table of doubled square roots, w_1, w_2, the corresponding counterpoise and beam weights respectively, C' the current given by the electrolysis, w'_1, w'_2, the counterpoise weight and beam weight applied, then we have

$$\frac{C^2}{C'^2} = \frac{w_1 d_1 + w_2 d_2}{w'_1 d_1 + w'_2 d_2}$$

where d_1, d_2 are constants. But $w_1/w_2 = w'_1/w'_2$; hence this equation gives

$$\frac{C^2}{C'^2} = \frac{w_1}{w'_1} = \frac{w_2}{w'_2}.$$

Thus w_1, w_2 are found by multiplying the ratio C^2/C'^2 by w'_1, w'_2 respectively, and the determination is complete.

When a very strong or a very weak current is required, as in the graduation of a hektoampere or a centiampere balance, it is desirable in the former case to allow the whole current to flow through the instrument, and only a convenient part through the electrolytic cell, and in the latter case to use a considerably greater current through the electrolytic cell than through the instrument. The current must therefore be divided in both these cases into two parts whose ratio is accurately known, and this may

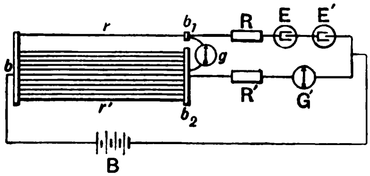

FIG. 41.

be done by the conductance bridge shown in Fig. 41. A set of parallel straight wires of platinoid are each soldered at one end to a thick terminal bar of copper b, and have soldered to them at the other ends thick terminal pieces of copper by which they can be connected in two groups by means of mercury troughs b_1, b_2. In the Figure they are shown in two groups of 10 and 1 respectively.

The wires are adjusted so that when they are in position they have all precisely the same resistance. (The method by which this equality is ensured is described in Chap. VIII. below.) Between the troughs b_1, b_2 a sensitive reflecting galvanometer (see p. 182 below) g is joined which indicates

no current when b_1, b_2 are at the same potential. The electrolytic cells E, E', and the instrument G to be standardized, are placed as shown in the figure when the standardizing current must be greater than that which the cells can carry, and the positions shown are interchanged when the reverse is the case. The currents are adjusted to balance in both cases by the rheostats R,R'. The currents are of course in the ratio of the conductances of the groups r, r' of the wires of the bridge.

CHAPTER VIII.

THE COMPARISON OF RESISTANCES.

WE give here some account of methods for the comparison of the resistances of conductors in which steady currents are kept flowing. In most cases the conductor to be compared is arranged in a particular way in connection with other conductors, which are then adjusted so as to render the current through a certain conductor of the system zero. From the known relation of the resistances of the other conductors the required comparison is deduced.

The form of galvanometer generally employed in the measurement of resistances is the reflecting galvanometer invented by Sir William Thomson, one arrangement of which is shown in Fig. 42. For most purposes the ordinary form of the instrument can be used. In this a mirror of silvered glass to which the needle-magnets are cemented at the back is hung within a cylindrical cell about half a centimetre in diameter. The ends of the cylinder are closed by glass plates from four to five millimetres apart, held in brass rings which can be screwed out or in so as to increase or diminish the length of the cell. The mirror is hung by a piece of a single silk fibre passed through a small hole in the cylindrical

surface of the chamber and fixed there with a little shellac. The mirror is only of slightly smaller diameter than the cylinder in which it hangs, so that in this

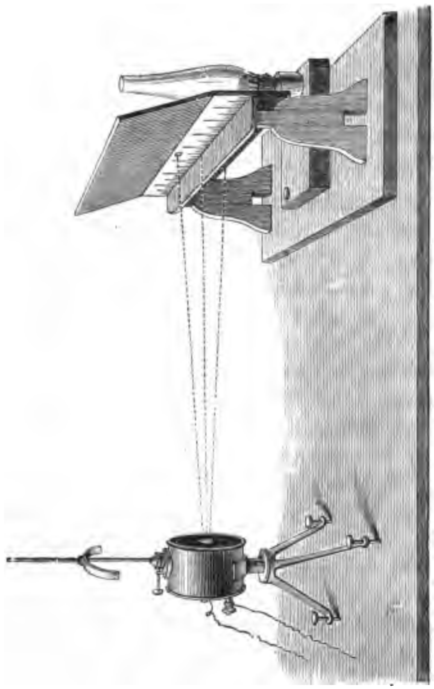

FIG. 42.

arrangement the fibre is very short, rendering it necessary in cases in which deflections have to be read off to allow for the effects of torsion. The cylindrical chamber is

screwed into one end of a cylinder of slightly greater diameter which fits the hollow core of the coil, and is called the galvanometer-plug. When the plug is in position the mirror hangs freely within its cell, with therefore the point of suspension on the highest generating line of the cylinder. Deflections of the needle are observed either by the Poggendorff telescope method, or, and much more generally, by the ordinary projection method described on pp. 7 and 8 above.

The weight of the needle and mirror is under one grain, and hence the period of free vibration of the suspended system about any position of equilibrium is short. The needle is also made to come quickly to rest by the smallness of the chamber in which it hangs. Since the mirror nearly fills the whole cross-section of the cell, the air damps the motion of the mirror to a very great extent even when the cell has its largest volume. The mirror may be made quite "dead-beat," that is, to come to rest without oscillation, by screwing in the front and back of the cell until the space is sufficiently limited.

In instruments in which it is desirable to avoid effects of torsion the galvanometer coil is made in two lengths which are fixed end to end, with a narrow space between them to receive the suspension piece. This piece forms a chamber in which the needle hangs between the two halves of the coil, and gives a length of fibre which at shortest is equal to the radius of the outer case of the coil, and which can obviously be made as long as is desired. The part of the hollow core at the needle is closed in front and at back by glass plates carried by brass rings. These can be screwed in or out by a key from without so as to diminish or increase the size of the chamber, and

thus render the needle system more or less nearly "dead-beat."

We shall suppose the galvanometer set up so that the deflections are read by the ordinary deflection method. It is only necessary to arrange that the needles when no current is flowing in the wires shall hang parallel to the plane of the coils. This is done as follows. A straight thin knitting wire of steel is magnetized and hung by a single silk fibre of a foot or so in length. This can easily be done by taking a sufficiently long single fibre of silk and forming a double loop on one end by doubling twice and knotting. In this double loop, made widely divergent, the steel wire is laid horizontally, and the single end of the fibre is attached to a support carried by a convenient stand, which is then placed so that the wire takes up a position in the direction of the horizontal component of the magnetic field where the needle is to be placed. A line can now be drawn parallel to the wire on the table beneath it. All that is necessary then is to place the galvanometer so that the front and back planes of the coil are vertical and parallel to this line, and adjust the lamp and scale as described above.

It is sufficient for our present purpose to state that if the needles be so small as in the Thomson reflecting galvanometer, and torsion can be neglected, the current in the coil may be taken as proportional to the tangent of the deflection angle, and therefore if that angle be not greater than three or four degrees the current may, with an error not greater than $\frac{1}{7}$ per cent., be taken as proportional to the deflection simply.

The galvanometer should be made as sensitive as possible by diminishing the directive force on the needle as

far as is practicable without rendering the needle unstable. This is easily done by placing magnets near the coil so that the needle hangs, when the current in the coil is zero, in a very weak magnetic field. That the field has been weakened by any change in disposition of the magnets, made in the course of the adjustment, will be shown by a lengthening of the period of free vibration of the needle when deflected for an instant by a magnet and allowed to return to zero. The limit of instability has been reached when the position of the spot of light for zero current changes from place to place on the scale, and the intensity of the field must then be slightly raised to make the zero position of the needle one of stable equilibrium.

Although not absolutely essential, except when accurate readings of deflections are required, it is always well when the field is produced by magnets, to arrange them so that the field at the needle is nearly uniform. It may therefore be produced by two or more long magnets placed parallel to one another at a little distance apart symmetrically with respect to the centre of the needle above or below it, and with their like poles turned in the same directions; or a long magnet placed horizontally with its centre over the needle, and mounted on a vertical rod so that it can be slided up or down to give the required sensibility, may be used.

Sensibility is sometimes obtained by the use of astatic galvanometers, but these are rarely necessary and are more troublesome to use than the ordinary non-astatic instrument.

For the comparison of the resistances of conductors other resistances the relations of which are known are employed. These are generally coils of insulated wire

wound on bobbins which are arranged so that the coils can be used conveniently in any desired combination.

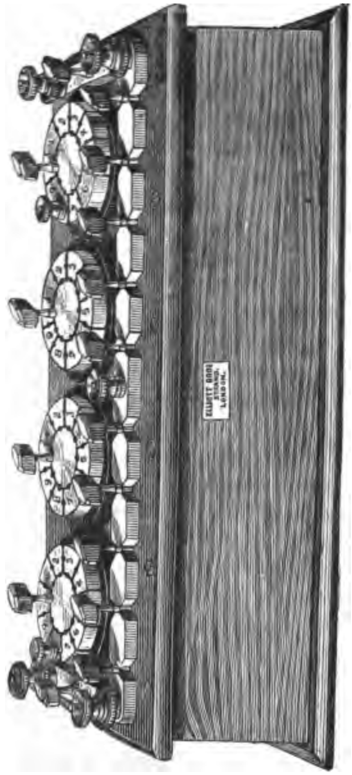

FIG. 43.

Such an arrangement of coils is called a resistance box. Figs. 43 and 44 show resistance boxes of different forms.

In a resistance box each coil has a separate core, which ought to be a brass or copper cylinder split longitudinally to prevent induction currents, and covered with thin rubber or varnished paper for insulation. These cores are shown in Fig. 45. The metallic core facilitates the cooling of the coil if an appreciable rise of temperature is produced by the passage of a current through it. After each layer of the coil has been wound it is dipped

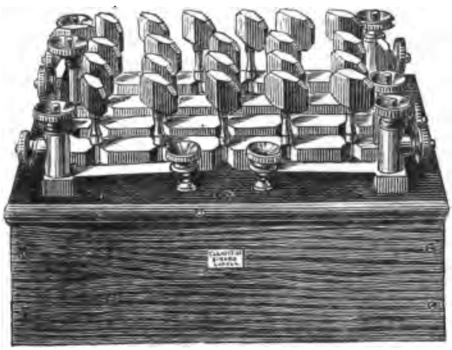

FIG. 44.

in melted paraffin, so as to fix the spires relatively to one another, preserve them from damp, and insure better insulation. It is of great importance to use perfectly pure paraffin, and especially to make sure that no sulphuric acid is present in it. Unless this precaution is observed trouble may be caused not only by the action of the acid on the metal of the conductor, but by the polarization effects due to electrolytic action in the acid paraffin.

Paraffin which is at all doubtful should be well shaken up when melted with hot water to remove the acid.

The wire chosen for the higher resistances is generally an alloy of one part platinum to two parts silver. This has a high specific resistance (p. 238 below) combined with a small variation of resistance with temperature. For the lower resistances wire of greater thickness is employed on account of its greater conductivity, which enables a greater length of wire to be used and thus facilitates accurate adjustment.

Coils are now sometimes made of "platinoid," a species of German silver which does not tarnish seriously with exposure to the air and has a low variation of resistance with temperature (see Table V.).

When a coil of given resistance is to be wound, a length of well-insulated wire of slightly greater resistance (determined by comparison at ordinary temperature by one of the processes to be described) is cut, doubled on itself at its middle point, and wound thus double on its core. This is done to avoid the effects of induction (see p. 206) when the current is in a state of variation, as when starting or stopping. After the coil has been wound its resistance is again measured, and if good insulation has been obtained, it ought now to show a slightly increased resistance, on account of the change produced in the wire by bending. The coil is fixed in position by two long brass or copper screws *d*, *d*, Fig. 45 passing through ebonite discs in the ends of its core, which fasten it to the cover of the box. These should be sufficiently massive to give no appreciable resistance. These screws are attached to two adjacent brass pieces, *a*, *a*, on the outside of the cover, and have the ends of the wire of the coil soldered to them so that the coil bridges across the gap

shown in the figure between every adjacent pair of brass pieces. The coil is now brought to the temperature at which it is to be accurate and finally adjusted so that its resistance taken between the brass pieces is the required resistance.

Coils are made in multiples of the " Ohm " or practical unit of resistance. The ohm is defined absolutely in

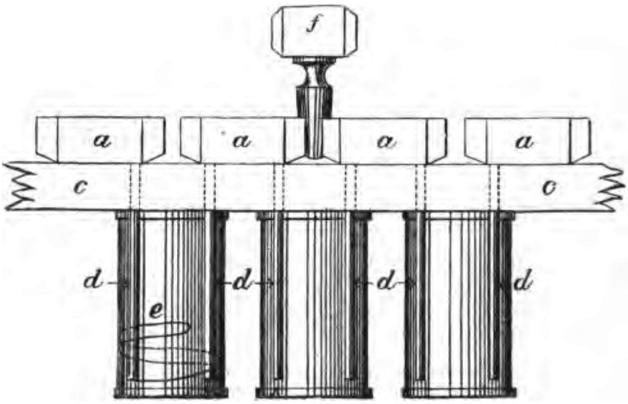

FIG. 45.

Chap. V. above: and it is there stated that the *Legal Ohm* as adopted by the International Congress of Electricians held at Paris in 1884 is equal to the resistance of a uniform column of pure mercury 106 centimetres long and one square millimetre in cross-section, at the temperature 0° C. Different forms in which copies of such a standard are made are described below, p. 243.

A series of coils are arranged in a resistance box in

some convenient order either in series or in multiple arc. Fig. 46 shows a series arrangement suitable for many purposes. The numbers indicate the number of ohms in the corresponding coils. The space between each pair of blocks is narrow above and widens out below, as shown in Fig. 45, to increase the effective

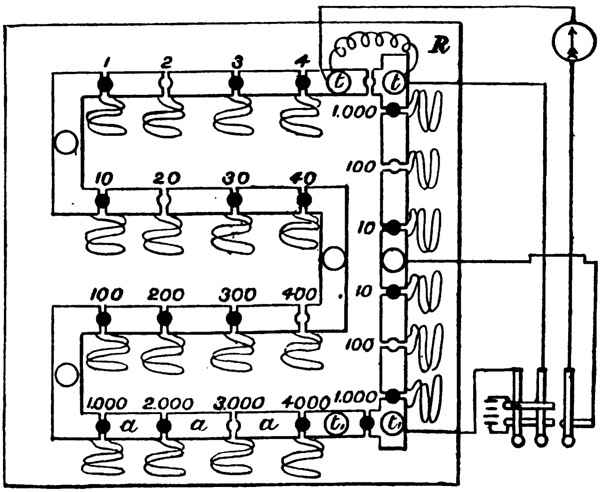

FIG. 46.

distance along the vulcanite from block to block. In the adjacent ends of the brass pieces, between which is the narrow gap, are cut two narrow opposite grooves, so as to form a slightly conical vertical socket. This fits a slightly conical plug, *f* in Fig. 45, which when inserted bridges over the gap by making direct contact between

the blocks, and when not thus in use is held in a hole drilled in the middle of the upper surface of the block. The coil is short-circuited when the plug is inserted, that is a current sent from one block to the other passes almost entirely across the plug on account of the much greater resistance of the coil. The handle, *f*, of the plug is generally made of ebonite.

The plan of arranging a series resistance box which is most economical of coils is a geometrical progression with common ratio 2. In such a box two units are generally provided to enable the box to be conveniently tested. The inconvenience of the arrangement is in the reduction of any resistance which it is proposed to unplug in the box to its expression in the binary scale of notation. For example if the resistance 370 is to be found on the box, this is expressed as $2^8 + 2^6 + 2^5 + 2^4 + 2$ or 101110010, and the corresponding plugs inserted, namely the first, fourth, fifth, sixth, and eighth of the plugs beyond the units. The process of reduction is performed as follows by dividing successively by 2, and writing the remainders as successive figures of the number from right to left in the order in which they are obtained, ending with the last quotient, which is of course 1.

2	370	
	185	0
	92	1
	46	0
	23	0
	11	1
	5	1
	2	1
	1	0

Hence 370 = 101110010 in the binary scale. It is not however always necessary to go through this process. Practice with a box on this principle leads soon to readiness in deciding what coils are to be unplugged, or what is the resistance of any set of coils which may be unplugged. It is well to remember that any coil of the series is greater by unity than the sum of all the preceding coils of the series.

The coils form a geometrical series from 1 to 4096 with a common ratio 2. The unit is duplicated for the reason stated above.

The "Dial" form of series resistance box shown in Fig. 43 above, is preferable to the ordinary forms for many purposes. It contains three or four or more sets of equal coils, each nine in number. One set consists of nine units, the next of nine tens, the next of nine hundreds, and so on. Besides these the box sometimes contains a set of nine coils each a tenth of a unit. Fig. 47 is a plan of a five-dial box. The sets of coils are arranged along the box in order of magnitude. Each set is arranged in series, and the blocks to which the extremities of the coils are attached are arranged in circular order round a central block, which can be connected to any one of the ten blocks of the set surrounding it, by inserting a plug in a socket provided for the purpose. Each central block, except the first and last, is connected by a thick copper bar inside to the initial block of the succeeding series of nine coils, as shown in Fig. 47 by the dotted lines. The ten blocks of each set of coils are numbered 0, 1, 2, . . . 9, as shown. Thus a current passing to one of the central blocks passes across through the bar to the next series of coils, then through the coils until it reaches a block connected to the central piece by a plug, when it passes across

to the centre and then to the next series of coils. If no coil of a series is to be put in circuit, the plug joins the central block to the coil marked zero.

In a five-dial box the central blocks are marked respectively TENTHS, UNITS, TENS, HUNDREDS, THOUSANDS, and the resistances are read off decimally at once. Thus supposing the centre in the first dial to be connected to the block marked 5, in the second dial to the block marked 7, in the third to that marked 6, the resistance put in circuit is 67·5 units.

The advantage of the arrangement consists in the fact that only one plug is required in each dial whatever the resistance may be, and since the plugs when no coils are included complete the circuit through the zeros, there is always the same number of plug contacts in circuit, instead of a variable number as in the ordinary arrangement.

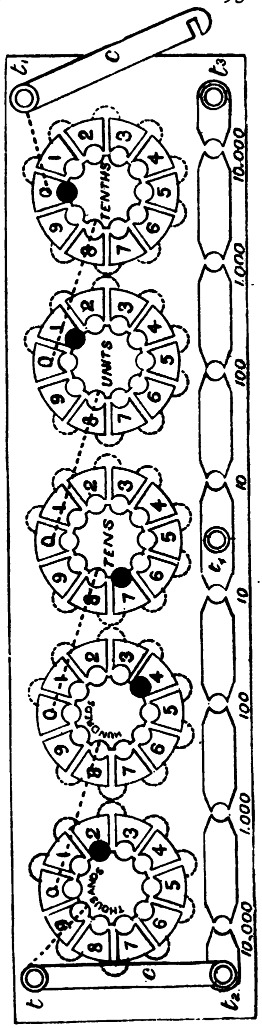

FIG. 47.

Besides the dial resistances there is generally in each box a set of resistances arranged in the ordinary way, and comprising two tens, two hundreds, two thousands, and sometimes two ten-thousands, fitted with terminals to allow the box to be conveniently used as a Wheatstone Bridge, as described below. The extremities of this series of resistances can be connected by means of thick copper straps with the series of dial resistances. Each pair of equal coils are sometimes wound on one bobbin to ensure equality of temperature.

It is sometimes desirable to have a ready means of varying the ratio of two resistances, or of increasing a single resistance by steps of any required amount. For this purpose a resistance slide is a convenient arrangement. A form devised by Sir William Thomson is shown at *CD* in Fig. 48. Along a metallic bar *r* in front of a series of equal resistance coils slides a contact piece *s* by which *r* is put in conducting contact with any one of the series of brass or copper blocks by which the coils are connected. The figure shows a combination of two slides used by Sir William Thomson and Mr. C. F. Varley for cable testing. Each resistance in *AB* is five times that of each coil in *CD*, and there is the same number in each, so that the whole resistance of *CD* is twice that of each coil in *AB*. The slider, *S*, of *AB* consists of two contact pieces insulated from one another on the slider, and at such a distance apart as to embrace two coils. The terminals of *CD* are connected to *CC* as shown in the figure, and therefore in whatever ratio the resistance *CD* is divided by the contact piece *s*, in that ratio is the joint resistance of the two coils *CC* divided. *CD* thus forms a vernier for *AB*. In the arrangement figured the resistance *CD* is divided into the two parts 12 and 8, and

therefore the sixth and seventh coils of AB which are between the terminals of S are divided into two similarly situated parts 12 and 8. Hence the whole resistance between A and B is divided into the two parts 56 and 44.

Fig. 49 shows a dial form of the double resistance slide. The main coils are on the left, the vernier coils on

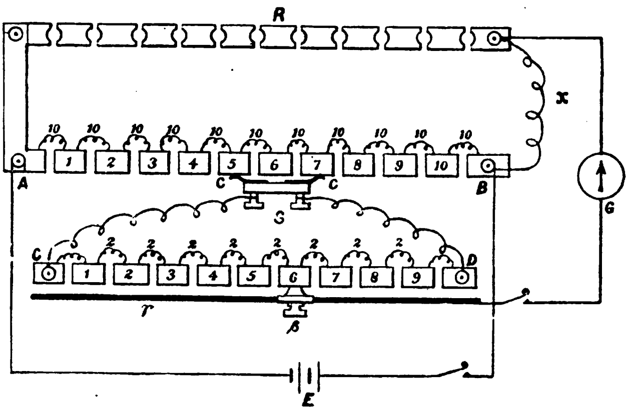

FIG. 48.

the right. Each slide may be detached and used independently if required.

Boxes in which the coils in circuit are in multiple arc were first made at the suggestion of Sir William Thomson, and called Conductivity Boxes, because the conductance * in circuit is obtained by adding the con-

* The word "Conductance" as stated above (p. 87) is now widely used instead of "Conductivity," and is also adopted in this book.

ductances of the coils. Fig. 50 shows the arrangement. Each coil is a resistance coil wound on a bobbin as

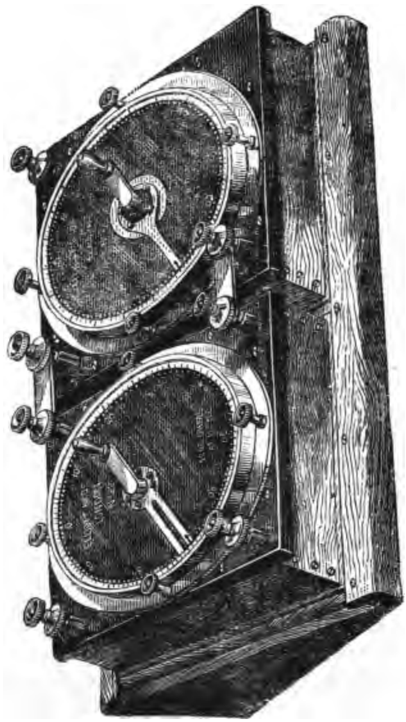

FIG. 49.

described above and has one extremity connected to a massive bar *a*, the other to a brass block *c*, outside the box, which can be connected by a plug to the massive

bar *b*. The resistance in circuit is obtained at once by adding the conductances of the coils thus in circuit, and taking the reciprocal of their sum. The conductances of the coils are marked on the corresponding blocks outside the cover.

This arrangement is very convenient for the measurement of low resistances such as one ohm and under, as it gives a long gradation of fractions by combination of the coils.

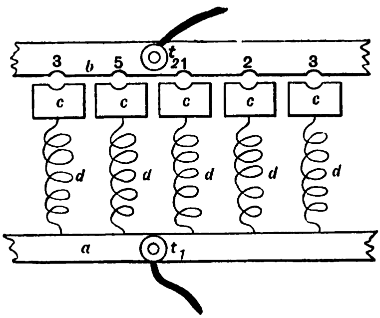

FIG. 5.

Sir William Thomson has proposed to call a box arranged thus a Mho-Box, where "Mho" is the word "Ohm" read backwards to indicate that the box gives conductances, that is reciprocals of resistances.

The resistance of almost all wires increases with rise of temperature, and the box is generally adjusted to be correct at a convenient mean temperature which is marked on the cover. The value of the resistance shown by the box at any other temperature is obtained when the change of temperature can be ascertained from the

known variation of resistance with temperature. A table of the variation of the resistances of different substances with temperature is given at the end of this volume.

The general internal temperature can be observed by means of a thermometer passed through one of the orifices which should be left in the side of the box to allow free circulation of air. Local changes of temperature may sometimes be produced in the coils without affecting appreciably the general internal temperature. These changes cannot be accounted for, as it is impossible to observe them with any accuracy, but can be avoided by using only the very feeblest currents, and continuing these for the shortest possible time.

The general internal temperature can also be measured by means of an auxiliary coil provided for the purpose. This is constructed of thick copper wire wound on ebonite, and extends along the whole length of the box. Since the variation of resistance of copper relatively to that of the wire of which the coils are constructed is known, we can by measuring the resistance of this auxiliary unit by the box itself obtain a closely approximate estimate of the internal temperature.

The temperature variation may be made for all the coils the same as the highest variation for any one, by introducing into each a piece of copper (conveniently at the bight after the coil is wound) just sufficient for the purpose.

In every case the blocks to which the coils are attached should be pierced with a socket for special plugs with binding terminals attached, by means of which any coil in the box may be brought into circuit itself. This is necessary for the testing of the box, which is done as

follows. In the case of the ordinary arrangement of coils Figs. 44, 46, each of the units is compared with a standard unit, then the two units together are tested against each of the 2s, then the 2s and a 1 are attested against the 5 and so on, until the 100s are reached. All the preceding coils put together give 100, which can be attested against each of the 100s, and this process is continued until the box is completely tested. The process can be checked by other possible combinations, and the whole of the results, if necessary, put together by the ordinary methods of combination.

If a dial box is to be tested the auxiliary unit, if it has one, suffices for the comparison of each of the units, then the nine units and the auxiliary unit give 10 for the comparison of each of the nine tens. These when compared give with the ten units 100 for the comparison of each of the hundreds, and so on.

In the case of a box arranged in geometrical progression with common ratio 2, and first term 1, the unit is duplicated for the sake of comparison. Each unit having been compared with a standard, they give together a comparison of the next coil, which is 2, then that with the two units give 4, with which the coil of 4 units can be compared, and so on.

The actual methods of comparing coils are described below (p. 219 *et seq.*). It is to be remembered that in the comparison of the coils of low resistance the connecting wires (which should be in all cases short and thick) must be taken into account.

In the use of a set of resistance coils it is important that the plugs be kept clean, and the ebonite top of the box, especially between the blocks of brass, kept free from dust and dirt. The ebonite may be freed from

grease by washing it with benzole applied sparingly by means of a brush, and a film of paraffin oil should then be spread over its surface. The plugs and their sockets may also be freed from adhering greasy films by washing in the same way with benzole or very dilute caustic potash. The latter should not however be allowed to wet the ebonite surface. If necessary the sockets may be scraped with a round-pointed scraper. On no account should the plugs or sockets be cleaned with emery or sand paper.

It is frequently necessary to adjust a current to a convenient strength by varying the amount of resistance in circuit. When the amount of resistance in circuit need not be known, this can be done most readily by means of a rheostat, or resistance coils in series with a rheostat, an arrangement which has the advantage of giving a continuous variation of the resistance. A form of rheostat constructed by Sir William Thomson is shown in Fig. 51. Two metal cylinders are mounted side by side on parallel axes and are geared so as to be driven at the same rate in the same direction by a third shaft turned by a crank. Along this shaft from end to end of the cylinder is worked a screw, which when turned moves a nut along a graduated scale at the top of the instrument. One of the cylinders is covered with well varnished paper, the other has a clean metal surface. A bare wire of platinoid or other material is wound partly on one cylinder partly on the other, and in passing from one cylinder to the other threads through a hole in the nut. Thus if the cylinder be turned the relative amounts of wire on the two cylinders can be varied at pleasure, and the wire is laid on them helically in a regular manner. The toothed wheel by which one of the cylinders is turned is con-

nected with the axle by means of a spring previously wound up so as to give a couple tending to wind the wire

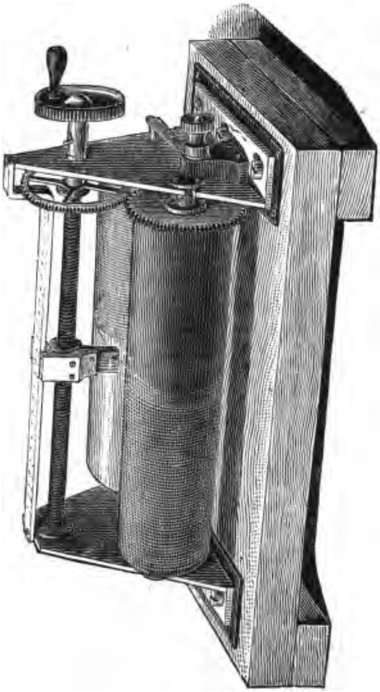

FIG. 51.

on the cylinder. The wire is thus kept taut and the spires prevented from shifting on the cylinder.

The course of the current is along the wire on the

paper covered cylinder, then to the bare cylinder. Thus the resistance in circuit is regulated by the amount of the wire on the former cylinder. A comparative estimate of this is given by the scale along which the nut moves.

The arrangement of screw and nut for guiding the wire above described seems to have been first used in a rheostat constructed by Mr. Jolin of Bristol.

A simpler form of rheostat, first used by Jacobi, consists of a single cylinder of insulating material round which the wire is wound in a helical groove. A screw of the same pitch as the groove is cut in the axle of the cylinder, and works in a nut in one of the supports. The cylinder, when turned, moves parallel to itself, so that the wire is kept in contact with a fixed rubbing terminal. Another rubbing terminal rests on the axle, to which one end of the wire is attached.

The method of comparing resistances of most general use is that known as Wheatstone's Bridge. The arrangement of conductors used is that shown in Fig. 52, with a battery, generally a single Daniell's or Menotti's cell, included in r_6, and a galvanometer in r_5. A much higher battery power is however sometimes required, especially in cable and other testing. The three conductors whose resistances are r_1, r_2 r_3 are coils of a resistance box provided with terminals so arranged that connections can be made at the proper places to form the bridge, for example as in Fig. 46, which shows a resistance box fitted up as a Wheatstone Bridge. It will be easy to make out in Fig. 52 the terminals corresponding to A, B, C, D respectively of Fig. 10. Fig. 44 above shows a so-called "Post-Office Resistance Box" in which the battery and galvanometer keys are mounted on the cover,

and permanently connected to the proper points inside the box; and Fig. 48 a Wheatstone Bridge arrangement of resistance slides.

The resistance to be compared is placed in the position *BD* (Fig. 52), and convenient values of r_1 and r_2 are chosen, while r_3 is varied until no current flows through the galvanometer. The value of r_4 is then found by (17) of Chapter VI, which since γ_5 is zero, may be written

$$r_4 = \frac{r_2}{r_1} r_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

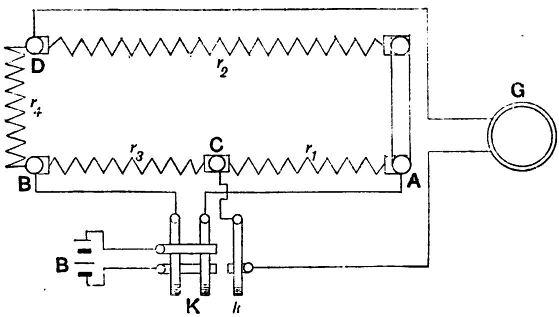

FIG. 52.

If r_1 and r_2 are equal, r_4 is equal to r_3, and is read off at once from the resistance box.

In the practical use of Wheatstone's Bridge we have generally to employ a certain battery and a certain galvanometer for the measurement of a wide range of resistances; and it is possible if great accuracy is required so to choose the resistances of the bridge as to make the arrangement have maximum sensibility. An approximate determination is first made of the resistance to be

measured. Call this r_4. It has been shown independently by Mr. Oliver Heaviside,* and by Mr. Thomas Gray,† that if the battery and galvanometer are invariable we should make

$$r_1 = \sqrt{r_5 r_6},\ r_3 = \sqrt{r_4 r_6 \frac{r_4 + r_5}{r_4 + r_6}},\ r_2 = \sqrt{r_4 r_5 \frac{r_4 + r_6}{r_4 + r_5}}.$$

If the resistances of the battery and galvanometer are at the disposal of the experimenter, then on the supposition that the resistance of the galvanometer may be taken equal to r_5, the most sensitive arrangement is that in which each of the resistances is equal to r_4.

Unless in particular cases in which great accuracy is necessary, any convenient values of r_1, r_2 will give results sufficiently accurate for all practical purposes; but in arranging the bridge with these the following rule should be observed: of the resistances r_5, r_6 of the galvanometer and battery respectively, connect the greater so as to join the junction of the two greatest of the four other resistances to the junction of the two least. This rule follows easily from (17) of Chap. VI. For interchanging r_5 and r_6 we alter only the value of D, and calling the new value D' we get

$$D' - D = (r_5 - r_6)(r_1 - r_4)(r_3 - r_2) \quad . \quad (2)$$

The expression on the right will be negative if $r_6 > r_5$ and r_1, r_3 be the two greatest or the two least of the other resistances. Hence on this supposition the value of D has been diminished, and therefore the current through the galvanometer for any small value of $r_2 r_3 - r_1 r_4$ increased by making r_6 join the junction of r_1, r_3 to that of r_2, r_4.

* *Phil. Mag.* vol. xlv. (1873), p. 114.

† *Ibid.* vol. xii. (1881), p. 283. For a proof of these results see the author's larger treatise, vol. i.

In cases in which the resistances in the bridge are large, a galvanometer of high resistance should also be used.

In the practical use of the method the electrodes of the battery should be carried to the terminals of a reversing key, so that the testing current may be sent in opposite directions if desired through the resistances of the bridge. Also a single spring contact-key, which makes contact only when depressed, should be placed in r_5. These keys are convenient when arranged side by side, so that the operator placing a finger on each can depress one after the other. A convenient form of wire rocker with mercury cups, combining the two keys, may be easily made by the operator. When the bridge has been set up and a test is about to be made, the single key in r_5 is first depressed to test whether any deflection of the galvanometer needle is produced without closing the battery circuit. If there is a deflection, this must be due either to thermoelectric action in the galvanometer circuit, or to leakage from the battery to the galvanometer wires. The procedure in this case will be stated presently. If there is no deflection, the operator then opens the galvanometer circuit, depresses the key which completes the battery circuit, and immediately after, while the former key is kept down, depresses also the galvanometer key. After the circuits have been completed just long enough to enable the operator to see whether there is any deflection of the needle, the keys are released so as to break the contact in the reverse order to that in which they were made. This order of opening the circuits enables him to make a second observation of deflection without its being necessary to again send a current. It is easy to imagine and construct a form of contact-making key, which being depressed a certain distance completes the

battery circuit, and on being depressed a little further completes the galvanometer circuit, and therefore on being released interrupts these circuits in the reverse order. This form of key is of use in the testing of resistance coils in which there is considerable self-induction. For general work, however, it is inconvenient, as the reverse order of making the contacts may have to be adopted. Again, in many practical operations, such as cable testing, &c., the contacts have to be made after different intervals of time in different cases.

The object of thus completing and interrupting the battery circuit before that of the galvanometer is partly to avoid error from the effects of *self-induction*. When a current in a conducting wire is being increased or diminished, an electromotive force, the amount of which depends on the arrangement of the conductor, is called into play, so as to oppose the increase or diminution of the current. The effect of this electromotive force is to produce, therefore, a weakening of the electromotive force of a battery for a very short time after the circuit is completed, and a strengthening during the very short interval in which the current falls from its actual value to zero at the interruption of the circuit. Its value is small when the wire is doubled on itself so that the two parts lie along side by side, the current flowing out in one and back in the other; but is very considerable if the wire is wound in a helix, and still greater if the helix contains an iron core. The electromotive force of self-induction is directly proportional to the rate of variation of the current in the circuit, and thus is explained the bright spark seen when the circuit of a powerful electro-magnet is *broken*.

If, then, one or more of the coils of a Wheatstone

Bridge arrangement were wound so as to have self-induction, the electromotive force thus called into play would, if the galvanometer circuit were completed before that of the battery, produce a sudden deflection of the galvanometer needle when the battery circuit is closed. All properly constructed resistance coils are, as has been stated, made of wires which have been first doubled on themselves and then wound double on their bobbins, and have therefore no self-induction. The wire tested, however, and the connections of the bridge have generally more or less self-induction, the effect of which, unless the contacts were made as described above, might be mistaken for those of unbalanced resistance. This mode of winding the coils also avoids direct electromagnetic effects on the coils on the galvanometer needle when the coils are placed near it.

If on depressing the galvanometer key at first as described above a current is found to be produced by thermoelectric or leakage disturbance, and the spot of light is therefore displaced, the operator keeping down the galvanometer key depresses the battery key, and observes if there is any permanent deflection of the spot of light from its displaced position during the time that the battery key is kept down. This is easily distinguished from the sudden deflection due to self-induction, as that immediately dies away to zero as the current rises to its permanent value.

If the sudden deflection of the galvanometer, as it may be in the case of a dynamo or electro-magnet, is too violent and long continued, the reversing key of the battery should be used, the battery contact made first in each case, and the mean of the results taken.

When comparing a resistance the operator first observes

the direction in which the mirror or needle is deflected when a value of r_3 obviously too great is used, and again when a much smaller value of r_3 is used. If the deflections are in opposite directions, the value of r_3, which would produce no deflection of the needle, lies between these two values, and the operator simply narrows the limits of r_3, until on depressing the galvanometer key no motion, or only a very small motion, of the needle is produced. It may happen, however, that the value of the resistance which is being compared may be between two resistances which have the smallest difference which the

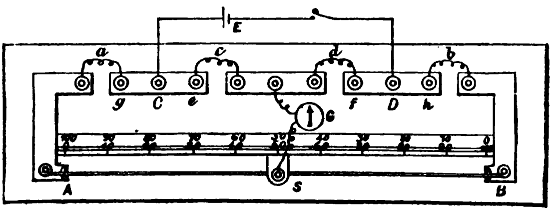

FIG. 53.

box allows. Thus with a resistance box by which with equal values of r_1 and r_2 he cannot measure less than $\frac{1}{10}$ of an ohm, he may either by making the ratio of r_1 to r_2, 10 to 1, or 100 to 1, obtain the values of r_4 to one or two places of decimals. Any inaccuracy in the relation of the arms of the bridge may be eliminated by reversing the arrangement, that is, interchanging r_1 and r_2, and r_3 and r_4, and taking the mean of the results.

Whatever be the ratio of r_1 to r_4, if he can read the deflections when first one and then the other value of r_3 (between which r_4 lies, and which differ by only $\frac{1}{10}$ of an

ohm) is used, he can find r_4 to another place of decimals by interpolation by proportional parts. For example, let the value 120·6 of r_3 produce a deflection of the spot of light of 6 divisions to the left, and 120·5 a deflection of 14 divisions to the right: the value of r_3 which would produce balance is equal to

$$120{\cdot}5 + {\cdot}1 \times 14/(14 + 6) = 120{\cdot}57.$$

The most accurate form of Wheatstone's Bridge is that introduced by Kirchhoff. In this an exact balance is obtained by moving a sliding contact-piece along a graduated wire which joins the two resistances r_1, r_4 of Fig. 52.

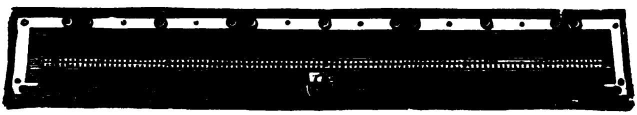

FIG. 54.

A diagrammatic sketch of the arrangement is shown in Fig. 53. S is the sliding-piece, A, B the wire along which it slides. A, B is stretched in front of a scale a metre in length graduated to half-millimetres and doubly numbered, from left to right and from right to left. The coils a, c, d, b of the diagram have the respective resistances r_1, r_2, r_3, r_4. Fig. 54 shows a form of the instrument manufactured by Messrs Elliott Bros.

Fig. 55 shows an easily-made and cheap form of wire bridge devised by Prof. T. Gray. w, w is the wire, made of platinoid or German silver, which is stretched above, but not in contact with, a base-board, passes round the insulating and supporting vulcanite block B from the

mercury cup c_1 to the other c_5. A vulcanite crossbar A clamps the wire in position near the cups. If the wire be long several such crossbars may be used. Each end of the wire is soldered to a large mass of copper, bent, as shown in Fig. 56, so as to dip into a mercury cup without any risk of contact of the mercury with the soldered junction. The cups should be of copper, and may conveniently be made of the form shown in the figure, and fixed in holes in the wooden or ebonite supporting-block. The ends of the copper pieces dipping into them should be carefully squared and bear against the copper bottoms.

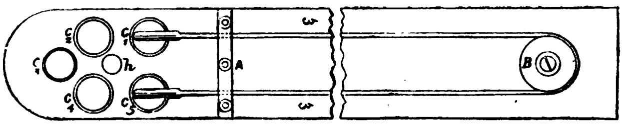

FIG. 55.

The wire is divided into parts of equal resistance by a process of calibration (p. 212 *et seq.*, below) and marks indicating these parts are made on a rule attached to the base-board, along which the contact-piece slides. A movable scale is used to subdivide the space between two divisions.

On a plate of ebonite or well-paraffined hard wood are fixed mercury cups c_2, c_3, c_4, made as just described. The auxiliary resistances r_1, r_4 of the bridge when required are placed between c_1 and c_2, c_5 and c_4, while the wires to be compared connect c_2 and c_3, c_3 and c_4. Since the wire w, w can be made long, the auxiliary resistances are not frequently required. When they can be dispensed with,

c_2 and c_4 are removed and c_3 placed in the socket *h*, and the wires to be compared are then placed between c_3 and c_1, c_3 and c_5.

A form of slide-wire bridge was used by Matthiessen and Hockin in the very careful comparisons of resistance made by them in their work as members of the British Association Committee on Electrical Standards; and it was found by these experimenters that an alloy of 85 parts of platinum with 15 parts of iridium formed an excellent material for the graduated wire. This alloy, they found, did not readily become oxidized. Platinum-silver alloy is however frequently employed.

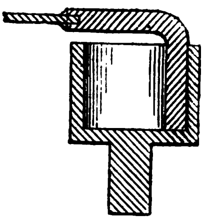

FIG. 56.

The contact piece is generally a well-rounded edge of steel with a slight notch to receive the wire. The knob pressed by the operator bends a spring which presses the contact piece with just sufficient pressure against the wire. A turning bar can be put into position to keep down the contact when desired. The sliding piece carries a vernier which enables fractions of a division to be read on the scale.

The method of testing by this instrument is precisely the same as by the ordinary Wheatstone Bridge, except

that when balance has been nearly obtained in the usual way, by varying the relation of the resistances r_1, r_2, r_3, for a particular position of the sliding piece, an exact balance is obtained by shifting the sliding piece in the proper direction along the wire. Supposing that the resistance of the wire per unit of length has been determined for different parts of the wire, and that the resistances of contacts have been determined (p. 223) and allowed for, the value of r_4 is at once found by taking into account the resistances of the segments of the wire AB on the two sides of the point, contact at which gives zero deflection.

The wire AB may be "calibrated" by one of the following methods. The first is that which was employed by Matthiessen and Hockin.* Let r_1 and r_4 (a, b in Fig. 53) be such resistances that balance is obtained at some point P in AB, with two coils r_2, r_3 (c, d in Fig. 53) differing in resistance by say $\frac{1}{10}$ per cent. Let $r_1 + a$ be the total resistance, including contacts, between C and P, and $r_4 + \beta$ that between D and P. Now alter r_1 by inserting a short piece of wire. This will shift the zero point along the wire through a certain distance to the left. Balance so as to find this point, which call P_1; then interchange r_2 and r_3, and balance again, and call the second point thus found P_2. Let z denote the resistance between P and P_1, z' the resistance between P and P_2, x the resistance of the short piece of wire added to r_1, and l the length of wire between P_1 and P_2. We have, neglecting connections of r_2, r_3,

$$\left.\begin{aligned} \frac{r_1 + a + x - z}{r_2} &= \frac{r_4 + \beta + z}{r_3} \\ \frac{r_1 + a + x - z'}{r_3} &= \frac{r_4 + \beta + z'}{r_2} \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

* *Reports on Electrical Standards*, p. 119.

from which we obtain for the resistance per unit of length between P_1 and P_2,

$$\frac{z - z'}{l} = \frac{r_2 - r_3}{l(r_2 + r_3)}(r_1 + r_4 + a + \beta + x) \quad . \quad (4)$$

The value of x is easily obtained with sufficient accuracy from either of equations (3), as z is approximately known from the known resistance of the whole wire. In this way the resistance per unit of length at different parts of the wire can be easily found, and, if necessary, a table of corrections formed for the different divisions of the scale.

Professor Carey Foster has given the following method for the calibration of the bridge wire. The arrangement is shown diagrammatically in Fig. 57. The battery shown in Fig. 53 is removed, and two equal copper bars are attached at C, D (Fig. 53), at right angles to the bars of the bridge at those points. Between the extremities of these is stretched a second slide wire. Or the slide wire of a second bridge, from which all other connections have been removed, may be connected to C and D by wires from the end bars to which it is attached. In place of the coils c, d of Fig. 53, and the middle bar of the bridge is substituted a single Daniell's or other cell. One terminal of the galvanometer is connected to a sliding piece on the wire W, the other to a sliding piece on the other wire, W'. In place of r_1 and r_4 are substituted two small resistances, one simply a piece of thick wire c, the other a resistance g, equal to that of a convenient portion say from 80 to 100 millimetres of the bridge wire. The former of these has been called the connector, the latter the gauge. They are connected to the bridge by mercury cups in the manner described

on p. 211 above, and some form of switchboard is usually employed to effect the interchanges described below.

Supposing the gauge placed first on the left and the connector on the right, the slide on W is moved close up to the extremity B, and balance is obtained by placing the slider on W' at some point near D. The gauge and connector are then interchanged, and balance is again obtained by shifting the slider on W' towards the left to some point b.

The gauge and connector are again interchanged, and balance obtained by shifting the slide on W to the left,

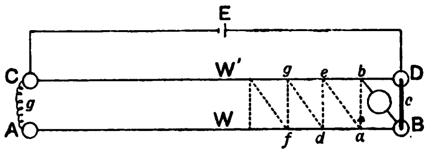

FIG. 57.

and so on until both wires have been traversed almost completely from end to end. The distance through which the slider is moved at each interchange of the resistance is read off, and gives, as we shall now show, a determination of the average resistance per unit of length over that portion of the wire. Let P and P' be points of contact on W and W' when balance is obtained, let the permanent resistances included with W, W' at the left-hand ends be denoted by a, b, and at the other ends by a', b' respectively, the resistance of the connector by c, of the gauge by g, of the wire from A to P by z, of the whole wire by w, of the wire W' from C to P' by z', and

of the whole wire by w'. If the connector be on the left and the gauge on the right, we have

$$\frac{c + a + z}{a' + z'} = \frac{g + b + w - z}{b' + w' - z'} \quad . \quad . \quad . \quad (5)$$

and if the gauge and connector be interchanged so that z receives a new value z_1,

$$\frac{g + a + z_1}{a' + z'} = \frac{c + b + w - z_1}{b' + w' - z'} \quad . \quad . \quad . \quad (6)$$

From these equations we get at once

$$g - c = z_1 - z. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

that is, the steps along W have each a total resistance equal to $g-c$, a result evident without calculation at all.

Again, supposing the gauge at first on the left, and next on the right, the slider on W' is shifted, and we get the equations

$$\frac{a' + z'}{g + a + z} = \frac{b' + w' - z'}{b + c + w - z}$$

$$\frac{a' + z'_1}{c + a + z} = \frac{b' + w' - z'_1}{b + g + w - z}.$$

These give

$$z_1 - z'_1 = (g - c)\frac{a' + b' + w'}{a + b + c + g + w} \quad . \quad . \quad (8)$$

The quantities on the right-hand side are all constants, and therefore the wire W' is thus divided into parts of equal resistance. From the known resistance of the whole wire, which can be found as shown on p. 354 below, the resistance of each part can be found. The steps on each wire are thus steps of equal resistance.

The following are the actual results obtained in the calibration of the slide-wire of a bridge performed in the Physical Laboratory of the University College of North Wales by the method just described.

Parts of the wire of equal resistance ($= r$).		Resistances of the parts included between the corresponding readings.	
Readings (zero taken at right hand end).	Lengths l.	Readings.	Resistance $= \frac{10r}{l}$.
0...10·59	10·59	0... 10	·94429 r
9·79...20·35	10·56	10... 20	·94697 ,,
19·70...30·26	10·56	20... 30	·94697 ,,
29·84...40·41	10·57	30... 40	·94607 ,,
39·69...50·22	10·53	40... 50	·94967 ,,
49·71...60·27	10·56	50... 60	·94697 ,,
59·80...70·35	10·55	60... 70	·94787 ,,
69·82...80·32	10·50	70... 80	·95238 ,,
79·86...90·38	10·52	80... 90	·95057 ,,
89·41...99·97	10·56	90...100	·94697 ,,
		0...100	9·47873 r

The numbers in the right-hand column are taken from tables. The results are of course not correct to the number of decimals given.

It will be noticed that the second reading in any line of the first column is not exactly the same as the first reading in the next line. This was caused through its being difficult to balance by adjusting the contact on the auxiliary wire. Balance was therefore obtained after a step was taken along the auxiliary wire by moving the slider through a short distance on the wire which was being calibrated.

The value of r found as described below, p. 220, was

·0452 ohm. From this the resistance of the part of the wire between two readings of the scale is found as shown in the table.

A modification of this method which works well in practice and avoids some difficulties has been made by Prof. T. Gray. The two wires *W*, *W'*, are arranged parallel to one another as in Fig. 58, and are connected at the ends *A*, *C* and *B*, *D* by two equal small resistances of suitable amount *g*. The equality of these resistances can be tested with great ease and delicacy by connecting

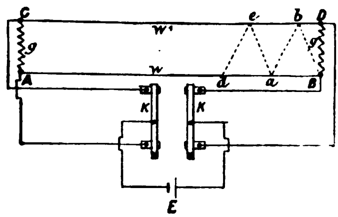

FIG. 58.

the battery at *A*, *B*, and balancing with the galvanometer between a point on *W* and another on *W'*, then interchanging the small resistances *g g* and observing if the balance is disturbed. If it is not the resistances are equal. When the resistances have been adjusted to equality, the battery is brought into contact at *A* and *D* and balance is obtained by placing one galvanometer terminal close to *B* on *W*, and the other at *b* on *W'*. The battery contacts are then transferred to *B* and *C*, and balance is obtained by shifting the terminal of the galvanometer on *W* to some point *a* while that on

W' is kept at b. The battery contact is then transferred to A, D, and balance obtained by moving the terminal on W' so that the points of contact are a, e, and so on.

The readings on the graduated scales are taken for the successive points of contact, and divide each wire, as will be shown presently, into steps each of resistance g.

The contact of the battery at A, D or B, C can be made by means of two simple rockers K, K, working between mercury cups or ordinary metal contacts, or by means of any simple key. This renders unnecessary any mercury cup switchboard arrangement for transferring coils.

Thus the method has the great advantage that the contacts are all permanent except those of the battery and the sliders, no one of which of course introduces any error.

Let contact be made by the battery at A and D, and balance be obtained with the galvanometer at points a and e on the wires W and W', then calling as before z, z' the resistances of the wires between A and a, C and e respectively, and w, w' the resistances of the whole wires, we have, neglecting (which will not affect the result) constant resistances of connecting bars, &c.,

$$\frac{w - z + g}{z} = \frac{w' - z'}{z' + g} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let the battery now be transferred to B and C and balance be obtained at d and e. Denoting the resistance between A and d by z_1, we again have

$$\frac{w - z_1}{z_1 + g} = \frac{w' - z' + g}{z'} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equations (9) and (10) give

$$z - z_1 = g\left(1 + \frac{w + g}{w' + g}\right) \quad . \quad . \quad . \quad . \quad (11)$$

or the steps along W are steps of equal resistance. The same can of course be proved for W'.

To avoid thermoelectric effects in such processes, the mean of the two positions of balance for opposite currents should always be adopted as the true position.

The slide-wire bridge may be used for the accurate comparison of resistance coils with a standard, say for the adjustment of single ohms with a standard ohm. Fig. 53 (p. 208 above) shows the arrangement adopted. r_1 and r_4 are the resistances of the coils a, b to be compared, and are nearly equal. r_2 and r_3 are the resistances of the two coils c, d, and are each nearly equal to r_1 or r_4. The connections are made by mercury cups as already described. Balance is obtained with the contact-piece somewhere near the middle of the slide-wire. The coils r_1, r_4, are then interchanged and balance again obtained. By (7) above we have

$$r_1 - r_4 = z_1 - z_2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where z_1, z_2 are the resistances of the wire from A to the point of contact in the two cases. If ρ be the resistance per unit of length for the whole wire, s_1, s_2 the distances (reduced, if necessary, by calibration, as shown above, to distances along a wire of uniform resistance ρ per unit of length) measured along the wire from A, we have

$$r_1 - r_4 = \rho(s_1 - s_2) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These results are evidently free from any uncertainty as to the resistance of the junctions of the slide-wire to the

copper bars at its ends, and from any error due to want of correspondence between the index mark on the sliding-piece and the point of contact.*

If a separate experiment be made with a coil of accurately known resistance r_1, just a very little less than that of the whole wire, and a second conductor of resistance r_4 so small that it may be neglected, the value of ρ may be obtained from the equation

$$\rho = \frac{r_1}{s_1 - s_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the coils compared are too unequal to allow balance to be made on the wire, a series of intermediary coils may be obtained, so as to give a gradual descent from one coil to the other.

The resistance of the wire between any two readings may also be determined by the following method, which is due to Mr. D. M. Lewis. The total resistance of the wire is approximately found by measuring it with an ordinary bridge consisting of a post-office set of coils or other available form of a resistance box. Two coils are then made, the resistance of each of which is less than unity by a quantity which is nearly equal to, but not greater than, the total resistance of the wire. These can be also made by means of an ordinary resistance box. Let R_1, R_2 be the as yet not accurately known resistances of these coils. Each is tested as follows in the slide wire bridge against a unit coil, a standard ohm for example. The unit coil is first placed in the position a of Fig. 53 and one of the two resistances, R_1 say, is placed in the

* The resistance of a coil may be accurately adjusted to any required value by first making it slightly too great, and then joining it in multiple arc with a thin wire cut so as to give as nearly as possible the required correction. If the observed resistance be r_4, and that required r_1, the resistance of the correcting wire is $r_1 r_4/(r_4 - r_1)$.

position *b*. The connections should be made by mercury cups as already described. In the positions marked *c*, *d* are placed permanently two coils of nearly equal resistance. The magnitudes of these need not be known, but should not be greater than one or two units. Balance is obtained with the slide S at a point near the end B of the slide wire, and the reading on the slide scale is taken. The coil R_1 and the unit are then interchanged, and balance obtained with the slide near A. The difference of the two readings gives the length of wire intercepted between them, and this must be equal in resistance to $1 - R_1$.

The other coil R_2 is now substituted for R_1 and two readings for which balance is obtained taken in the same way. These give a length of the wire the resistance of which is $1 - R_2$.

The two resistances are now put together in series and tested against the unit in precisely the same way, and give between the two readings taken a length of wire of resistance $R_1+R_2 - 1$.

Now from a previously made calibration of the wire the resistances of the three portions of the wire thus observed can be obtained in terms of the resistance of the calibration-step, and three equations are thus available for the determination of the three unknown quantities R_1, R_2, and r, the resistance of the step used in calibration, as in p. 216 above. The following table gives the results of this process applied to the slide wire the calibration of which is given above.

Positions of the Resistances.		Readings on Slide Wire.	Resistances between these readings in terms of r. Obtained from Table, p. 216 above.
Left.	Right.		
R_1 1	1 R_1	1·40 97·72	9·131 [9·47875 − ·13220 − ·21590 = 9·13065r]
R_2 1	1 R_2	0·14 98·97	9·368 [9·47873 − ·01322 − ·09754 = 9·36797r]
$R_1 + R_2$ 1	1 $R_1 + R_2$	69·70 31·45	3·625 [3·79058 − ·02843 − ·13717 = 3·62498r]

Here $1 - R_1 = 9{\cdot}131r, \; 1 - R_2 = 9{\cdot}368r$

$$R_1 + R_2 - 1 = 3{\cdot}625r$$

and therefore

$$r = \frac{1 - R_1}{9{\cdot}131} = \frac{1 - R_2}{9{\cdot}368} = \frac{R_1 + R_2 - 1}{3{\cdot}625} = \frac{1}{22{\cdot}124} = {\cdot}0452.$$

Substituting this value of r in the first two equations we find R_1 and R_2. This can be used, as shown at p. 216 above, to find the resistance of the portion of the wire between any two readings of the scale.

An accurate comparison of two nearly equal resistances, for example a unit with its copy, can be obtained by making r_2 and r_3 to be compared occupy the positions c, d, of Fig. 53. Balance is first obtained with r_2 and r_3 in one pair of positions, then they are interchanged and balance again obtained. Assuming that the permanent resistances are included in r_1, r_4, r_2, r_3, and giving z_1, z_2 the same meanings as at p. 219 above, we have

$$\frac{r_2}{r_3} = \frac{r_1 + z_1}{r_4 + w - z_1} = \frac{r_4 + w - z_2}{r_1 + z_2}$$

$$= \frac{r_1 + r_4 + w + z_1 - z_2}{r_1 + r_4 + w - (z_1 - z_2)}$$

and therefore

$$\frac{r_2 - r_3}{r_3} = \frac{2(z_1 - z_2)}{r_1 + r_4 + w - (z_1 - z_2)} \quad . \quad . \quad (15)$$

Hence the greater $r_1 + r_4$ the greater $z_1 - z_2$. Thus by choosing a pair of resistances as nearly equal as possible, and sufficiently great, r_2 and r_3 may be compared to any needful degree of accuracy.

The permanent resistances, α, β say, corresponding to the coils a, b of Fig. 53, may be estimated by the following method, by which two low resistances can be measured when the ratio of two others is accurately known. Let the resistances r_2, r_3 of c, d in Fig. 53 have the known ratio μ. We shall suppose r_1 and r_4 to be so low resistances that, with a value of μ differing considerably from unity, balance can be found on the wire. Balance is obtained with the coils in the positions c, d, shown in Fig. 53; then r_2 and r_3 are interchanged, and balance is again obtained. We have

$$\mu = \frac{r_1 + z_1}{r_4 + w - z_1} = \frac{r_4 + w - z_2}{r_1 + z_2}.$$

From these equations we obtain

$$r_1 = \frac{z_1 - \mu z_2}{\mu - 1}, \quad r_4 = -w + \frac{\mu z_1 - z_2}{\mu - 1} \quad . \quad (16)$$

If thick copper pieces be substituted for the coils a, b of Fig. 53, their resistances, if the connections as is understood are made with proper mercury cups, may be taken as zero, and α and β are approximately given by (16). The values of α, β thus obtained may be used for the correction of the values of r_1, r_4 found as just described. This correction will not be appreciably affected by the

unknown permanent resistances corresponding to the coils c, d, if r_2, r_3 are taken moderately large so that the actual ratio may be taken as equal to their known ratio.

Neither of the arrangements of Wheatstone's Bridge described above is at all suitable for the comparison of the resistances of short pieces of thick wire or rod, for example, specimens of the main conductors of a low resistance electric light installation, the resistances of which are so small as to be comparable with, if not less than, the resistances of the contacts of the different wires by which they are joined for measurement. To obtain an accurate result in such a case, we must compare, directly or indirectly, the difference of potential between two cross-sections in the rod which is being tested, with the difference of potential between two cross-sections in a standard rod, while the same current flows in both rods, in a direction parallel to the axis at and everywhere between each pair of cross-sections.

Sir William Thomson has so modified Wheatstone's Bridge, by adding to it what he has called *secondary conductors*, as to enable it to be used with all the convenience of the ordinary arrangement, for the accurate comparison of the resistance of a foot or two of thick copper conductor with that between two cross-sections in a standard rod. The arrangement is shown in Fig. 58. CD are two cross-sections, at a little distance from the ends of the conductor to be tested, and AB are two similar cross-sections of the standard conductor. These rods are connected by a thick piece of metal, so that the resistance between B and C is very small, and the terminals of a battery of low resistance are applied at the other extremities of the rods as shown. The sections B, C are connected also by a wire BLC, and the sections A, D by a

wire *AMD*, in each case by as good metallic contacts as possible. *BLC* and *AMD* may very conveniently be wires, along which sliding contact-pieces *L* and *M* can be moved, with resistances *R*, *R*, *R*, *R* of half an ohm or an ohm each, inserted as shown in the figure. The sections *A*, *D* are so far from the ends of the rods, and the wires *AMD*, *BLC* are made of so great resistance (one or two ohms is enough in most cases), that the current throughout the portions of the conductors compared is parallel to

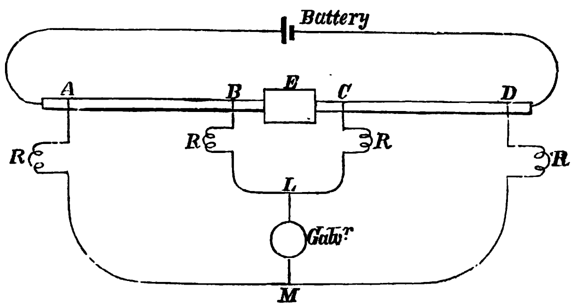

FIG. 59.

the axis, and the effect of any small resistance of contact there may be at *A*, *B*, *C*, *D* is simply to increase the effective resistance of *BLC* and *AMD* by a small fraction of the actual resistance of the wire in each case. The terminals of the galvanometer *G* are applied at *L* and *M*, and the circuits of the galvanometer and battery are completed through a double key as in the ordinary bridge. A reversing key is inserted in the battery circuit as in other cases, to enable the comparison to be made with both directions of current.

Let the resistances AM, DM be denoted by r_1, r_3; BL, CL by a, b; AB, CD by r_2, r_4; and BC by s. Suppose r_1

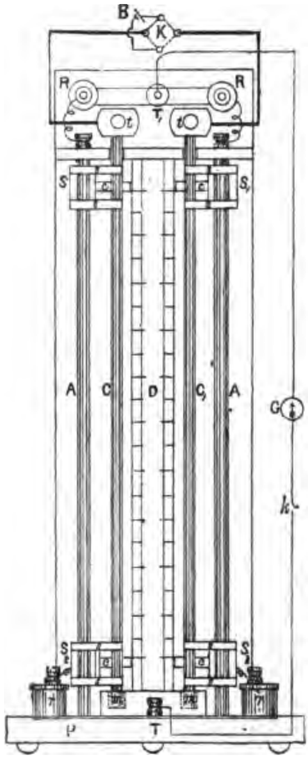

FIG. 60.

and r_3 to be varied by moving the sliding piece at M till no current flows through the galvanometer. To find the

relation which must hold among the resistances when this is the case, we may suppose the point L connected by a bar of zero resistance, with the cross-section of E which is at the same potential as L. Call this cross-section K. The resistance of the portion of BC to the left of K is $as/(a+b)$, and the portion of to the right $bs/(a+b)$. The resistance between B and KL is therefore $\{a^2s/(a+b\}/\{a+as/(a+b)\}$, or $as/(a+b+s)$, and similarly that between C and KL is $bs/(a+b+s)$. Hence by (1) we have

$$r_3\left(r_2+\frac{as}{a+b+s}\right)=r_1\left(r_4+\frac{bs}{a+b+s}\right),$$

or

$$r_1r_4-r_3r_2=\frac{s}{a+b+s}(ar_3-br_1)\ .\quad .\quad (17)$$

Now s has been supposed very small in comparison with $a+b$, and a and b can be easily chosen so as to make ar_3-br_1 approximately equal to zero. Hence equation (4) reduces to

$$r_1=\frac{r_3r_2}{r_4}\quad .\ .\ .\ .\ .\ .\ .\quad (18)$$

the formula found above for the ordinary Wheatstone Bridge.

The apparatus illustrated in Fig. 60 is convenient for the carrying out of this method in practice. On a massive sole plate of iron, P, are mounted two vertical guide-rods of copper, A, A, and parallel to these the rods to be compared, viz., a standard rod C, and the rod to be tested C_1. C, C_1 are supported with their lower ends in two mercury cups cut in a single block of copper. This block corresponds to the piece E in Fig. 60. The upper ends

of C, C_1 are fixed in screw blocks of copper, t, t, to which also are attached the terminals of a constant battery B of low resistance. A reversing key K is interposed between t, t and the battery. A scale D graduated along its two edges nearly fills the space between the rods C, C_1.

A pair of resistance coils, r, r, are fixed to the sole plate, and have one terminal of each connected by a strip of copper, which also carries the terminal screw T. The other terminals of these coils are fixed to two copper slides, S_2, S_3, which move along, but are insulated from, the guide-rods, and carry contact pieces c, c, each of which is bevelled off to a knife-edge on a level with its upper side. This knife-edge is pressed against the corresponding rod by springs s, s, which are insulated so as not to touch the rods. The coils r, r are attached directly to the contact pieces c, c. Thus $S_2\, r\, T\, r\, S_3$, corresponds to the partial circuit $BRLRC$ of Fig. 60.

Near the upper ends of C, C_1 is a similar arrangement of sliders, S, S_1 with spring contacts and attached coils, R, R. These coils are connected by a copper strip which carries the terminal T_1. The coils R, R are attached to the upper ends of the guide-rods A, A, and through these to the sliders S, S_1. The guide rods are so thick that no appreciable change is made in the ratio of the resistances of the parts of the partial circuit SRT_1RS_1 on the two sides of T_1 by varying the positions of the sliders. This partial circuit corresponds to $ARMRD$ of Fig. 60.

Each pair of coils, r, r and R, R, may be wound on a single bobbin with advantage. The arrangement is thereby rendered more compact, and there is less risk

of error from difference of temperature between the bobbins, or of thermoelectric disturbance between their terminals.

Between T and T_1 is placed the galvanometer G, which is provided with a simple key k, placed for convenience in the actual arrangement beside the reversing key K.

In the use of the instrument the rods to be compared are placed in position, and the sliders on the rod of lower resistance are placed so that their upper edges, and therefore their knife-edges, are opposite the lowest and uppermost divisions of the scale. The lower contact piece on the other rod is placed with its upper edge opposite the lowest division of the scale on that side. The upper contact piece on the same rod is then shifted until no current flows through the galvanometer. Balance is obtained for both directions of the current, and the mean position of the slider taken, to eliminate error from thermoelectric disturbance.

A number of standard rods of different thicknesses are provided with the instrument in order that nearly equal ratios may be obtained over a wide range of low resistances.

The following method was used for the same purpose by Messrs. Matthiessen and Hockin in their researches on alloys. AB, CD, Fig. 61, are the two rods to be compared. They are connected in circuit with two coils of resistances r, s, which have between them a graduated wire WW', as in Kirchhoff's bridge. SS' are two sharp knife-edges, the distance of which apart can be accurately measured, fixed in a piece of dry hard wood or vulcanite, and connected with mercury cups on its upper side. This arrangement is placed on the conductor AB, so that the knife-edges making contact include between them a length

SS' of the rod. TT' is a precisely similar arrangement placed on CD. One terminal of the galvanometer is applied at S, and the resistances r, s adjusted so that a point P on the wire which gives balance is found for the other terminal. The terminal of the galvanometer is shifted to S', and a second point P' found by varying the resistances of the coils from r_1, s_1 to r'_1, s'_1 in such a manner as to keep the sum $r+s$ constant. Similarly

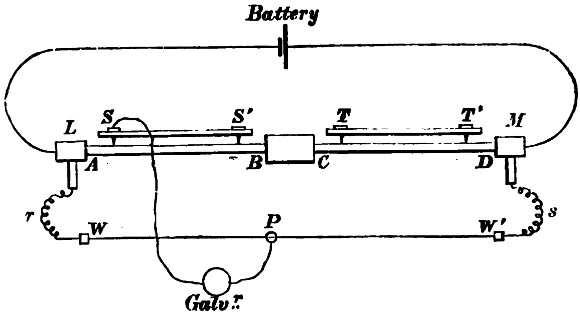

FIG. 61.

balance is found for TT' with values r_2, s_2, r'_2, s'_2, for the resistances of the coils, fulfilling the condition that the sum $r+s$ is the same as in the former case. Let a, b, c, d, k denote the resistances between L and S, L and S', L and T, L and T', L and M respectively; α, β, γ, δ the resistance between W and P in the four cases, κ the resistance of the whole wire WW'. We have by (1)

$$\frac{a}{k-a} = \frac{r_1+\alpha}{s_1+\kappa-\alpha},$$

and therefore

$$\frac{a}{k} = \frac{r_1 + \alpha}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $R = r + s + \kappa.$

Similarly

$$\frac{b}{k} = \frac{r'_1 + \beta}{R}$$

Therefore

$$\frac{b - a}{k} = \frac{r'_1 - r_1 + \beta - \alpha}{R} \quad . \quad . \quad . \quad (20)$$

In the same way we get

$$\frac{d - c}{k} = \frac{r'_2 - r_2 + \delta - \gamma}{R}. \quad . \quad . \quad . \quad (21)$$

and combining the last two equations we get for the ratio of the resistances of the conductors between the pairs of knife edges—

$$\frac{b - a}{d - c} = \frac{r'_1 - r_1 + \beta - \alpha}{r'_2 - r_2 + \delta - \gamma} \quad . \quad . \quad . \quad (22)$$

The following method of comparing resistances is in principle the same as Thomson's Bridge with secondary conductors, and Matthiessen and Hockin's method described above, as, like them, it consists in comparing the difference of potential between two cross-sections near the ends of the conductor to be tested with the difference of potential between two cross-sections in a standard conductor, when the same uniform current is flowing in both. It is, however, more readily applicable in practice, and is very useful for a great many purposes, as for example, in the testing of the armatures or magnet coils of machines, in the estimation of the resistances of contacts, and in the determination of the specific con-

ductivities of thick copper wires or rods. All that is required is a small battery, a suitable galvanometer of sufficient sensibility, and two or three resistance coils of from $\frac{1}{5}$ ohm to 1 ohm. These coils may very conveniently for many purposes be made of galvanized or tinned iron wire of No. 14 or 16 B.W.G., wound round a piece of wood $\frac{1}{2}$ inch thick, from 8 to 10 inches broad, and from 12 to 18 inches long, with notches cut in its sides, at intervals of a quarter of an inch, to keep the wire in position. To avoid any electromagnetic effect which may be produced by the coils if they happen, when carrying currents, to be placed near the galvanometer, the wire should be doubled on itself at its middle point, the bight put round a pin fixed near one end of the board, and the wire then wound double on the board, the two parts being kept far enough apart to insure insulation. Resistance coils made in this way are exceedingly useful for electric-lighting experiments, as the thickness of the wire and its exposure everywhere to the air prevent undue heating by strong currents, or, if there is much heating, obviate the risk of damage. For the battery a single cell, as for example a gravity-Daniell, or, if the battery is to be carried from place to place, two hermetically sealed chloride of silver cells, which may be joined in series or in multiple arc as required, may very conveniently be used. As instrument of comparison Sir William Thomson's centiampere balance used as voltmeter or his old form of graded potential galvanometer* is convenient for many practical purposes; but when very great

* That is a high resistance galvanometer in which the needle system, or magnetometer, can be placed with its centre at different distances from the centre of the coil to give different degrees of sensibility, and further provided with one or more magnets to intensify the magnetic field at the needles when required. See *Theory and Practice of Abs. Meas. in El. and Mag.*, Vol. ii.

accuracy is aimed at, as when the method is used for the measurement of the (specific) conductivity of short lengths of thick metallic wires by comparison with a standard, a sensitive reflecting galvanometer of resistance great in comparison with that of the conductor between the points at which the terminals are applied should be employed, and the battery should be of as low internal resistance as possible.

The galvanometer is first set up and made of the requisite sensibility either by adjusting, as described in p. 185 above, the intensity of the field in which it is placed, or, if it is a graded galvanometer, by placing the magnetometer at the position nearest the coil, and dispensing with the field-magnet.

The conductor whose resistance is to be compared, and one of the coils whose resistance is known, are joined in series with the battery. It is advisable to have this circuit at a distance of a few yards from the galvanometer, so that accidental motions of the wires carrying the current may not have any sensible effect on the needle. One operator then holds the electrodes of the galvanometer so as to include between them, say, first the wire which is being tested, then the known resistance, then once more the wire being tested, in every case taking care not to include any binding screw connection, or other contact of the conductors. The known resistance should, when great accuracy is required, be so chosen that the readings obtained in these two operations are as nearly as may be equal.

Let the mean of the readings for the first and third operations be V scale divisions, for the second V'; let r denote the known resistance, and x the resistance to be found.

Since by Ohm's law the difference of potential between any two points in a homogeneous wire, forming part of a circuit in which a uniform current is flowing, is proportional to the resistance between those two points, we have

$$x = \frac{V}{V'} r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The resistance of a contact of two wires whether or not of the same metal may be found in the same manner, by placing the galvanometer electrodes so as to include the contact between them, and comparing the difference of potential on its two sides with that between the two ends of a known resistance in the same circuit. Care must however be taken in all experiments made by this method, especially when the galvanometer circuit includes conductors of different metals, to make sure that no error is caused by thermal electromotive forces. To eliminate such errors the observations should be made with the current flowing first in one direction and then in the other in the battery circuit.

The following results of some measurements of the resistance of a Siemens S D_2 dynamo machine, made on May 4, 1883, in the Physical Laboratory of the University of Glasgow, may serve to illustrate this method. An iron wire coil, of half an ohm resistance, was joined to one of the terminals of a standard Daniell, and short wires attached to the other terminal of the cell and the free end of the coil were made to complete the circuit through the armature, by being pressed on two diametrically opposite commutator bars, from which the brushes and the magnet connections had been removed. The

electrodes of the galvanometer, which was one of Sir William Thomson's dead-beat reflecting galvanometers of high resistance, were applied alternately to the same commutator bars, and to the ends of the half ohm, and the readings recorded. The following are the results, extracted from the Laboratory Records, of three consecutive experiments:

EXPERIMENT I.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	857	643
,, ,, armature	597	383

EXPERIMENT II.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on armature	607	393
,, ,, ½ ohm	874	660
,, ,, armature	607	393

EXPERIMENT III.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	874	660
,, ,, armature	607	393
,, ,, ½ ohm	872	658

The first experiment gives for x the value, $383 \times \cdot 5/643$, or ·298 ohm. The other two experiments, although their numbers are different, give very nearly the same result, which agrees closely with a measurement made about eight months before, by the same method, with a graded potential galvanometer.

In the ordinary testing of the armatures of machines by this method, the circuit of the battery may be completed through the brushes; but if the machine has been wound on the shunt system, care must be taken to previously disconnect the magnet coils. In every case

the galvanometer electrodes must be placed on the commutator bars directly.

Prof. Tait* has used a differential galvanometer (see below, p. 238) for this method of determining low resistances. The conductors to be compared were arranged in series, so that the same current flowed through both. The terminals of one coil were then placed at two points on one conductor, the terminals of the other coil at two points on the other, such that the galvanometer deflection was zero. The difference of potential between the points

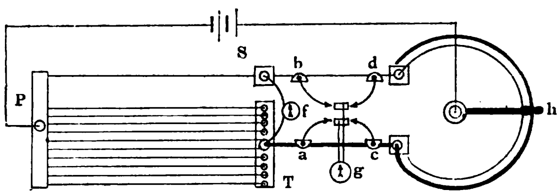

FIG. 62.

of each pair was therefore the same in the two cases. Hence the lengths of portions of the two conductors of equal resistance were obtained.

The following zero method, due to Prof. T. Gray, is founded on the same principle. The arrangement of apparatus is shown in Fig. 62. One terminal of a battery of one or two low resistance cells is attached to a stud on a thick copper bar *P*, the other terminal to a metallic axis round which the copper bar *h* turns. The bar *h* makes contact at its outer end with a bare wire and a bare rod bent round into concentric circles with centre at the axis of the bar, and having a pair of remote extremities connected with mercury cups or binding terminals, and the

* *Trans. R.S.E.*, vol. xxviii. 1877-8.

other pair of extremities free as shown. To one of these terminals is connected one end of the bar to be tested, to the other one end of the standard bar. The other end of one of these bars, say the standard, is connected to a mercury cup S, which is in line with, but is insulated from, a row of mercury cups or a mercury trough cut in a copper bar placed parallel to P. Between this bar and the trough are stretched a series of parallel wires all of the same material and length and as nearly as possible of the same resistance; and a single wire, of the same resistance, material, and length, connects the bar P and the cup S with which the standard bar is in contact. These wires may be conveniently straight rods of platinoid, an eighth of an inch in diameter, and six feet long, soldered at one end to the bar P, and at the other to stout well-amalgamated copper terminals dipping into the mercury cups or trough. The wires may be made of the same resistance by means of a slide wire bridge, or by the method described below.

The cup S and the terminals T are now brought to one potential by turning the bar h round on the circular wire until a sensitive galvanometer, f, joining them shows no deflection. This galvanometer is then left connected, and by means of a second sensitive galvanometer, g, two pairs of points a, d and c, d are found between which in each case no current flows when they are connected by a wire. Each pair of points are therefore at the same potential. Hence if we denote by r_1 the resistance of the standard between b and d, by r_2 that of the other rod between a and c, and by n the number of wires joining P and T, we have

$$r = \frac{r_1}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

A differential galvanometer* with two independent pairs of terminals may be employed for this method. One coil may be made to join a, b, the other c, d, or one coil may be made to join b, d, and the other a, c. In the former case either the effect on each coil must be made zero, or care must be taken to connect the terminals to a, b and c, d so that the magnetic effects of the two coils at the needle may be opposed. The resistance of the galvanometer coils, except when the current in each coil is made zero, must be so great as not to cause any sensible alteration of the potentials at the points at which the terminals are applied.

The wires joining P to S and T may be tested for equality as follows. Two nearly equal wires are made to join P to S and P to T, and h is placed so that the galvanometer f shows zero current. The wire joining P to T is then removed and another put in its place. If the current in f still remain zero for the same position of h the latter wire and the former are of the same resistance. If not the necessary correction is made (see footnote p. 220) and the comparison repeated.

In order that the conducting powers of different substances may be compared with one another, it is necessary to determine their *specific resistances*, that is, the resistance in each case of a wire of a certain specified length and cross-sectional area. We shall here define the specific resistance of any substance at any given temperature as the resistance between two opposite faces of a centimetre cube of the material at that temperature.†

* For an account of this instrument and its use in the measurement of resistance see Maxwell's *Electricity and Magnetism*, Vol. i., or the author's *Theory and Practice of Absolute Measurements*, &c., Vol. ii.

† The reciprocal of this (called below the specific conductivity) may be advantageously called the *electric conductivity* of the substance, if the word conductivity be set free by the general adoption of the word *conductance* for the reciprocal of the resistance of a given conductor.

This resistance has been very carefully determined for several different substances at ordinary temperatures by various experimenters, and a table of results is given below (Table V.).

To measure the specific resistance of a piece of thin wire, we have simply to determine the resistance of a sufficiently long piece of the wire by the ordinary Wheatstone Bridge method described above, and from the result to calculate the specific resistance. Let the length of the wire be l cms., its cross-section s square cms., and its resistance R ohms. Then the specific resistance of the material would be Rs/l ohms. The length l is to be carefully determined by an accurately graduated measuring-rod; and the area s may be found with sufficient accuracy in most cases by direct measurement, by means of a decimal wire gauge measuring to a hundredth of a millimetre. If, however, the wire be very thin, the cross-section may, if the density is known, be accurately obtained in square cms. by finding the weight in grammes of a sufficiently long piece of the wire (from which the insulating covering, if any, has been carefully removed), and dividing the weight by the product of the length and the density. Very thin wires are generally covered with silk or cotton, which may very easily be removed, without injury to the wire, by making the wire into a coil, and gently heating it in a dilute solution of caustic soda or potash. The coating must not in any case be removed by scraping.

If the density is not known, it may be found by weighing the wire in air and in water by the methods described in books on hydrostatics. All the weights, from 1 gramme upwards, ordinarily used in weighing are made of brass, and hence when conductors of nearly the same specific

gravity as brass are weighed in air, the correction for buoyancy may be neglected. The weighing in water however must be corrected both for expansion of water with rise of temperature and for the weight of air displaced by the weights. For a temperature of 13°C. these corrections are as follows:—for expansion of water an increase of loss of weight in water of ·059 per cent.; for buoyancy of air a diminution of apparent weight in water of about ·0143 per cent. Care should be taken in weighing to prevent air bubbles from adhering to the sides of the specimen; and the water used for weighing should first have been carefully boiled to expel the air contained in it. All error of this kind may be avoided by boiling the water with the specimen in it, and then allowing both to cool together.

If the wire be thick, and a sufficient length of it to render possible an accurate measurement of its resistance by the ordinary bridge method is not conveniently available, one of the methods of comparing small resistances described above (p. 224 *et seq.*) is to be used. The most convenient in many cases is that described in p. 231—236, in which the resistance between two cross-sections of the bar to be tested is compared with that between two cross-sections of a standard rod of pure copper. The cross-sections should, if the distance between them be not thereby made too small, be chosen so as to make the two resistances nearly equal. If we put V for the number of divisions of deflection on the scale of the potential galvanometer, when the electrodes of the galvanometer are applied to the standard rod, at cross-sections l cms. apart; V' that when they are applied to the rod being tested, at cross-sections l' cms. apart, then we have for the ratio of the resistance of unit length of the wire tested

to the resistance of unit length of the standard at the temperature at which the comparison is made, the value $V'l/Vl'$. If s and s' be the respective cross-sectional areas, which in this case are easily determinable by measurement with a screw-gauge, and we assume that the temperature at which the measurements of resistance are made is 0° C., we get for the ratio of the specific resistances at 0° C. the value $V'ls'/Vl's$, and therefore also for the ratio of their specific conductivities $Vl's/V'ls$. This last ratio multiplied by 100 gives the percentage conductivity at 0° C. of the substance as compared with that of pure copper. If, as will generally be the case, the temperature at which the experiments are made be above the freezing-point, the value of $100Vl's/V'ls'$, may be taken as the percentage of the specific conductivity of pure copper at the observed temperature possessed by the substance, and this, if the wire tested is a specimen of nearly pure copper, will be nearly enough the same at all ordinary temperatures.

If in experiments by this method the electrodes are applied by hand to the conductors, the operator should, especially if the electrodes and the conductors tested are of different materials, be careful not to handle the wires, but should hold them by two pieces of wood in strips of paper passed several times round the wires, or by some other substance which conducts heat badly, so that no thermal electromotive force may be introduced into the circuit of the galvanometer (see above, p. 205). He may conveniently make the galvanometer contacts by means of two knife edges fixed in a piece of wood which can be lifted from one conductor to the other without its being necessary to handle the galvanometer wires in any way. This will besides render any measurement of the length

of the conductor intercepted between the galvanometer electrodes unnecessary, as l is equal to l'. We have then for the percentage specific conductivity of the substance the value $100 Vs/V's'$.

As an example of this method we may take the following results of a measurement (made in the Physical Laboratory of the University of Glasgow) of the specific conductivity of a short piece of thick copper strip. The specimen was joined in series with a piece of copper wire of No. 0 B.W.G. of very high conductivity, in the circuit of a Daniell's cell of low resistance. The electrodes of a high resistance reflecting galvanometer applied at two points 700 cms. apart in the copper wire gave a deflection of 153·5 divisions, when applied at two points 500 cms. apart in the strip 270 divisions. The weight of the wire per metre was 443 grammes, of the strip per metre 186·3 grammes. Hence the specific conductivity of the copper strip was 96·6 per cent. of that of the wire against which it was tested.

The accurate realization of a standard ohm, as defined on p. 72 above, involves the determination of the specific resistance of mercury, an operation which requires great care and considerable experimental skill. This determination has been made by several experimenters, among others by Lord Rayleigh and Mrs. Sidgwick and by Messrs. Glazebrook and Fitzpatrick at Cambridge, and by Messrs. Hutchinson and Wilkes at Baltimore.*

The value obtained by Lord Rayleigh and Mrs. Sidgwick for the resistance at 0° of a column of mercury

* An account of Lord Rayleigh and Mrs. H. Sidgwick's determination will be found in *Phil. Trans. R.S.*, Part I., 1883, or the author's *Theory and Practice*, &c., vol. i. For the later determinations of Messrs. Glazebrook and Fitzpatrick and Messrs. Hutchinson and Wilkes see *Phil. Trans. R.S.*, 1888, and *Phil. Mag.* July, 1889.

1 metre long and 1 square millimetre in cross-section was ·95412 B.A. unit (p. 71 above). Messrs. Glazebrook and Fitzpatrick's value for the same resistance is ·95352 B.A. unit, Messrs. Hutchinson and Wilkes found it to be ·95341 B.A. unit. Previous measurements made by Werner Siemens and Matthiessen gave ·9536 B.A. unit and ·9619 B.A. unit respectively for this resistance. It will be noticed that the mean value for this resistance given by the three later measurements quoted lies between these, but much nearer to the former. Messrs. Siemens Brothers for a long time used the resistance of a column of mercury specified as above as the unit of resistance, and standard units were issued by them to experimenters. One of these examined by Lord Rayleigh gave ·95365 B.A. unit for its resistance at the temperature 16·7° at which it was stated to be correct.

Standard ohms have been made in mercury, by using tubes bent so that the requisite length is obtained in a compact form, but they are not very convenient in use, and are of course liable to breakage. A copy of the standard ohm can however be easily made when the resistance of a column of mercury of definite cross-section and length has been accurately found. Figs. 63 and 64 show such copies. Fig. 63 is the usual form of the standard. It is made of platinum-silver wire, wound within the lower cylinder. The space within up to the top of the wider cylinder is filled with paraffin wax. The ends of the coil are attached to two thick electrodes of copper rod, bent as shown and kept in position by a vulcanite clamp. The ends of these when the coil is used are placed in mercury cups in the manner already explained, and should always, before the coil is placed in position, be freshly amalgamated with mercury. The lower cylinder

up to the shoulder is placed in water when the coil is in use, and the temperature of the water is ascertained by means of a thermometer in the hollow core of the cylinder. The variation of the resistance of the coil with temperature is known, and hence its resistance at any observed temperature can be obtained. Of course care must be taken not to expose the standard to too strong currents, and to keep the temperature as near as possible to the

FIG. 63.

normal temperature at which the standard is given as correct.

Fig. 64 shows a form of the standard constructed by Messrs. Elliott and Co. according to a suggestion made by Professor Chrystal. A thermoelectric couple, of which one junction is within and close to the coil, and the other outside the case, is used to determine the temperature of the coil. In the form in which the instrument is now made the external junction is not brought out through

the bottom of the case as shown, but the wire is brought out at the top of the case, and then joined to a wire of the other metal which is entirely outside and attached to one of the binding screws. The external junction is of course placed in water the temperature of which is measured, and the thermal current is observed by means of a galvanometer connected to the terminals. This gives the difference of temperatures between the junctions and therefore the temperature of the coil.

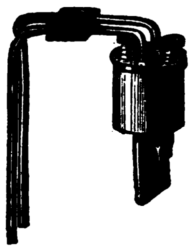

FIG. 64.

On account of the uncertainty of the temperature of the coil, and its liability to loss of insulation by deposition of moisture on the upper surface of the cylinder, Prof. J. A. Fleming* has constructed a standard in which the case containing the coil is a hollow circular ring of brass made by screwing together by projecting flanges two square sectioned circular troughs. The electrodes (rods arranged as in Fig. 63) proceed to the ring through two upright brass tubes from 5 to 6 inches in length, from which they are insulated by vulcanite collars at the bottom, and at the top by two vulcanite funnels corrugated on the outside,

* *Phil. Mag.* Jan. 1889.

and projecting above the tubes. Paraffin oil placed in these vulcanite funnels prevents loss of insulation by condensation of moisture on the insulating pieces.

The measurement of a very high resistance such as that of a piece of insulating material cannot be effected by means of Wheatstone's Bridge, and recourse must be had in most cases to electrostatic methods in which the required resistance is deduced from the rate of loss of charge of a condenser, the plates of which are connected by the substance in question. If, however, the resistance of the material be not too great, and a large well-insulated battery of from 100 to 200 cells, and a very sensitive high resistance galvanometer are available, the following method is the most convenient. First join the galvanometer, also well insulated, and the resistance to be measured (prepared as described below, p. 25, to prevent leakage) in series with as many cells as gives a readable deflection, which call D. Now join the battery in series with the galvanometer alone, and reduce the sensibility of the instrument to a suitable degree by joining its terminals by a wire of known resistance, and, to keep the total resistance in circuit great in comparison with the resistance of the battery, insert resistance in the circuit. Let E and B denote respectively the electromotive force and resistance of the whole battery, G the resistance of the galvanometer, S the resistance joining its terminals in the second case, R the resistance introduced into the circuit of the galvanometer in that case, and X the resistance to be found; we have for the difference of potential between the terminals of the galvanometer in the first case the value,

$$\frac{EG}{G+B+X} = mD. \quad . \quad . \quad . \quad . \quad (25)$$

where m is the factor by which the indications of the galvanometer must be multiplied to reduce them to volts. In the second case the resistance between the galvanometer terminals is $SG/(S+G)$, and therefore the difference of potential between them is,

$$\frac{E\dfrac{SG}{S+G}}{B+R+\dfrac{SG}{S+G}} = \frac{ESG}{(B+R)(S+G)+SG} = mD \quad (26)$$

Hence combining equations (25) and (26) so as to eliminate E and m, and solving for X, we get,

$$X = \frac{D_1}{D}\left(B+R+G+\frac{(B+R)G}{S}\right) - (B+G) \;. \quad (27)$$

If X be great in comparison with the remainder of the resistance in circuit the term $(B+G)$ may be neglected.

This method was used by Mr. T. Gray and the author for the determination of the specific resistances of different kinds of glass. The specimens of glass were in the form of thin, nearly spherical flasks about 7 cms. in diameter, with long narrow and thick walled necks. The thin walls of the flask were brought into circuit by filling it up to the neck with mercury, and sinking it to the same level in a bath of mercury, then joining one terminal of the battery to the external mercury by a wire passed down the long neck, and the other to the mercury in the bath without. This mercury bath was an iron vessel contained in a sand-bath which could be heated to any required temperature. A well-insulated galvanometer (constructed by aid of a grant from the Government Research Fund to a special

design*) of high resistance and great sensitiveness was included in the current. A battery of over 100 Daniell's cells was used, and after a reading of the galvanometer in one direction had been taken and recorded, with the corresponding temperature of the glass, the coatings of the flask were connected together until the next reading was about to be taken. For this the current was reversed, and the deflection taken as before, and so on. The "electric absorption" was thus reversed between every pair of readings, and lasted in most cases about three minutes. The resistances were therefore those existing after three minutes' electrification. The result for the glass of highest insulation tested, which was lead glass of density 3·14, was a specific resistance at 100° C. of about 8400×10^{10} ohms. The resistance was halved for each 8·5° or 9° rise of temperature.

A modification of this method for which a potential galvanometer or voltmeter is very suitable, may be used for the determination of the insulation resistance of the conductors in an electric light installation.

The conductors are disconnected from the generator and both ends from one another. They are then joined at one end by the potential galvanometer in series with a battery of as many cells as gives a readable deflection. The number of divisions corresponding to this deflection is read off, and the number of divisions which the battery gives when applied to the galvanometer alone is then observed. Call the latter number V and the former V'; and let E divisions be the total electromotive force of the battery. Let the resistance of the battery which may be determined by the method described below (p. 258) be B ohms, the resistance of the galvano-

* *Proc. R.S.* vol. xxxvi (1884).

meter G ohms, and the insulation resistance to be found R ohms ; we have plainly,

$$V = \frac{EG}{B + G}, \quad V' = \frac{EG}{B + G + R}.$$

Therefore,

$$R = (B + G)\left(\frac{V}{V'} - 1\right) \quad . \quad . \quad . \quad . \quad (28)$$

If B be small in comparison with G we may put

$$R = G\frac{V - V'}{V'} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

A shunt-wound generating machine giving sufficient electromotive force may be used instead of the battery, and in this case R is found by equation (29).

The insulation resistance for unit of length is found from this result by *multiplying* by the length of either of the conductors.

This method is applicable to the measurement of the insulation resistance of cables or telegraph lines, but for details the reader is referred to the manuals of testing in connection with these special subjects.

In the case of insulating substances the method just described requires the use of so powerful a battery that it is quite inapplicable except when the specimen, the resistance of which is to be measured, can be made to have a large surface perpendicular to the direction of the current through it, and of very small dimensions in that direction. Such a case is that of the insulating covering of a submarine cable in which the current by which the insulation-resistance is measured flows across the covering between the copper conductor and the salt water in which the cable is immersed.

In general, therefore, in the determination of the insulating qualities of substances which are given in comparatively small specimens it is necessary to have recourse to the electrometer method mentioned above (p. 246), of which we shall give here a short account.

The most convenient instrument for this purpose is Sir William Thomson's Quadrant Electrometer. For a full description of this instrument, and a detailed account of the mode of using it, see Chap. V. above. The electrometer, having been carefully set up according to the most sensitive arrangement, and found to be otherwise in good working order, is tested for insulation. One pair of quadrants is connected to the case according to the instructions for the use of the instrument, and a charge producing a difference of potential exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected for an instant to the electrometer-case, the other at the same time to the electrode of the insulated quadrants, and the percentage fall of potential produced in thirty minutes or an hour by leakage in the instrument is observed. If this is inappreciable, the instrument is in perfect order. For practical purposes the insulation is sufficiently good when the same battery being applied to charge the electrometer alone as is applied to charge the cable, or condenser formed as described below, there is not a more rapid fall of potential without the cable or specimen than with it; for there can then be no error due to leakage.

An air condenser, well insulated by glass stems varnished and kept dry by pumice moistened with strong sulphuric acid, is adjusted to have a considerable capacity, and its insulated plate is connected to the insulated quad-

rants of the electrometer, and the other to the electrometer-case, to which the other pair of quadrants is also connected. A charge producing as great a difference of potential as before is given to the condenser and electrometer thus arranged, and the fall of potential observed by means of the electrometer. If the loss in a considerable time be also inappreciable, the condenser insulates properly, and its resistance may be taken as infinite.*

The specimen of material to be tested is now placed so as to connect the plates of the condenser. The manner in which this is to be done of course depends on the form of the specimen. If it is a flat sheet, it may be covered on each side, with the exception of a wide margin all round, with tinfoil, and thus made to form itself a small condenser which is to be joined by thin wires in multiple arc with the large condenser. The edges and margins of the sides of the specimen should be carefully cleaned and dried, and covered with a thin coating of paraffin to prevent conduction along the surface between the two tinfoil coatings, when the condenser is charged. It is advisable, when possible, to coat the whole surface including the tinfoil with paraffin, and to make the contacts with the tinfoil plates by means of thin wires also coated with paraffin for some distance along their length from the tinfoil. The plate condenser thus formed should be supported in a horizontal position on a block at the middle of the lower surface. The upper coating is made the insulated plate.

If the specimen be cup-shaped, as, for example, if it be

* A condenser of any other kind, such as those used in cable testing, the insulating material between the plates of which is generally paper soaked in paraffin, may be used instead of an air condenser, but as the resistance of the latter may, if the glass stems be well varnished and kept dry, be taken as infinite, and there is besides no disturbance from the phenomenon called *electric absorption*, it is preferable to use an air condenser if possible.

in the usual form of an insulator for telegraph or other wires, the hollow may be partially filled with mercury, and the cup immersed in an outer vessel containing mercury, so that the mercury stands at nearly the same level outside and inside. The lip of the cup down to the mercury on both sides is to be cleaned and coated with paraffin, as before, to prevent leakage across the surface. A thin wire connected with the insulated plate of the condenser is made to dip into the mercury in the cup, and a similar wire connected with the other plate of the condenser dips into the mercury in the outer vessel. Strong sulphuric acid may, on account of its drying properties, be used with advantage instead of mercury as here described, when the substance is not porous and is not attacked by the acid.

In every case in which, as in these, the insulating substance and the conductors making contact with it form a condenser of unknown capacity, the condenser used in the experiment must be arranged to have a capacity so great that the unknown capacity thus added to it, together with the capacity of the electrometer, may be neglected in the calculations.

The condenser, if it has been disconnected, is again connected as before to the electrometer. One electrode of a battery of from six to ten small Daniell's cells in good order, is also connected with the electrometer case, and the other electrode is brought for a short time, thirty seconds say, or one minute, into contact with the insulated plate of the condenser at any convenient point, such for example as the electrode of the electrometer connected with the insulated pair of quadrants. The battery electrode is then removed, and the condenser and electrometer left to themselves.

The condenser has thus been charged to the potential of the battery, which will be indicated by the reading on the electrometer scale at the instant when the battery is removed. The deflection of the electrometer needle will now fall, more or less slowly according to the insulation resistance of the condenser with its plates connected by the material being tested. Readings of the position of the spot of light on the electrometer scale are taken at equal intervals of time, and recorded, and this is continued until the condenser has lost a considerable portion, say half, of its potential.

The resistance of the insulating material is easily calculated from the results in the following manner. Let V be the difference of potential between the plates of the condenser at any instant, Q the charge of the condenser at that instant, which may be taken as proportional to the deflection on the electrometer scale, and C its capacity (Chap. xi.). We have $Q = CV$, and therefore $dQ/dt = CdV/dt$. But $-dQ/dt$ is the rate of *loss* of charge, that is, the current flowing from one plate to the other, and this is plainly equal by Ohm's law to V/R. Hence $-dQ/dt = V/R$ and therefore

$$C\frac{dV}{dt} + \frac{V}{R} = 0.$$

Integrating we get,

$$\log V + \frac{t}{CR} = A \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where A is a constant. If V be the difference of potential t seconds after it was V_0, we get by putting $t = 0$ in (28) $A = \log V_0$. Hence (28) becomes

$$\frac{t}{CR} = \log \frac{V_0}{V}$$

and
$$R = \frac{t}{C} \frac{1}{\log \frac{V_0}{V}} \text{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

If $V = \frac{1}{2} V_0$, we have $R = t/C \log 2$.

If the condenser have a resistance so low as to add materially to the rate of discharge, an additional experiment must be made in the same way to determine the resistance of the condenser alone, with its plates connected only by its own dielectric. Let R_c denote the resistance of the condenser, determined by equation (29) from the results of the latter experiment, and R_i the resistance of the specimen; by equation (12) (p. 87) $1/R = 1/R_i + 1/R_c$, and therefore

$$R_i = \frac{RR_c}{R_c - R_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If C has been obtained in C.G.S. electrostatic units of capacity, it may be reduced to electromagnetic units by dividing by the square of the number of electrostatic units of capacity equivalent to the electromagnetic unit, that is (see Chap. XI.) by 9×10^{20} nearly.

When an air condenser is used, its capacity can generally be obtained approximately by calculation from the dimensions and area of the plates. For example, if two parallel plates of metal, placed at a distance d apart, very small in comparison with any dimension of either surface, have a difference of potential V, and there be no other conductor or electrified body near, it can easily be shown that the capacity on a portion of area A near the centre

* It is to be remembered that the logarithms to be here used are Naperian logarithms. The Naperian logarithm of any number is equal to the ordinary or Briggs' logarithm multiplied by 2·302585.

of either plate is $A/4\pi d$. Hence in the example below, we have for the capacity of the disk of area A the value $A/4\pi d$, if we neglect the non-uniformity of the electrical distribution near the edge.

If C has been taken in absolute C.G.S. electromagnetic units of capacity (see Chap. XI.), we obtain R from (29) in cms. per second,* which may be reduced to ohms by dividing by 10^9.

When a condenser such as one of those used in submarine telegraph work is used, the capacity C, of which is known in microfarads,† then since a microfarad is $1/10^{15}$ C.G.S. electromagnetic units of capacity, we have for R in ohms the formula

$$R = 10^6 \frac{t}{C} \frac{1}{\log \frac{V_0}{V}} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The following are results actually obtained in tests of a specimen of insulating material made in the form of an ordinary telegraph insulator. An air condenser consisting of two horizontal brass disks, the distance of which apart could be regulated by means of a micrometer screw, was joined with the insulator made into a small condenser with mercury inside and outside, as described above. The lower disc was of considerably greater diameter than the upper, which had a diameter of 12·54 cms., and the distance between them was adjusted to be 1 cm. The upper disk was connected to the insulated pair of quadrants, and the lower to the electrometer case. Calling A the area of the upper plate, and d the distance between them, we have, neglecting the effect of the edges of the

* In the electromagnetic system of units a resistance has the dimensions of velocity. See Chap. XI.

† See Chap. XI. below.

upper disk, for the capacity of this condenser the value $A/4\pi d$ in C.G.S. electrostatic units. Hence in the actual case $C = 9{\cdot}828$. The interior surface of the insulator covered by the mercury was so small, and the thickness of the material so great, that, even allowing the material to have a high specific inductive capacity, the capacity of the condenser which it formed was small in comparison with that of the air condenser. The experiment gave, when the condenser and insulator were joined as described, $V_o = 251$, $V = 100$, $t = 5640$ seconds. Hence,

$$R = \frac{5640}{9{\cdot}828 \times 2{\cdot}303 \times \log_{10}\frac{251}{100}} = 623,$$

in seconds per centimetre (C.G.S. electrostatic units of resistance). As the condenser was not insulating perfectly, a separate test was made for it alone, with the results $V_o = 239$, $V_1 = 182$, $t = 6120$. Hence

$$R_e = \frac{6120}{9{\cdot}828 \times 2{\cdot}303 \times \log_{10}\frac{239}{182}} = 2286,$$

and therefore by (30)

$$R_i = \frac{623 \times 2286}{2286 - 623} = 857,$$

in seconds per centimetre.

Multiplying this result by 9×10^{20} (the approximate value of v^2, see Chap. XI.), to reduce to electromagnetic units, we get for the resistance of the insulator 7712×10^{20} cms. per second, or 771×10^{12} ohms.

We shall now consider very briefly the measurement of the resistance of a battery. This term is not perfectly

definite in meaning, as there is reason to believe that the resistance as well as the electromotive force of a battery depends to some extent on the current flowing through the battery, and further the resistance and the electromotive force, and possibly also the polarization of the battery are affected by differences of temperature. But the information which in practice we generally require from the test, is really what available difference of potential can be obtained with a certain working resistance in the external circuit. This could be obtained at once by connecting the terminals of the battery by this resistance, and measuring the difference of potential by means of a quadrant electrometer or a potential galvanometer. If we call this difference of potential V, and the electromotive force of the battery when on open circuit E, then putting R for the external resistance we may write

$$\frac{E}{R+r} = \frac{V}{R} = \gamma \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where r is a quantity the definition of which is simply that it satisfies this equation. If the battery had the same electromotive force E, when generating the current γ, as when on open circuit, then r would be the effective resistance of the battery; but, although this is not the case, we may without being led into error still speak of it as the resistance of the battery for the current γ. In fact, the value of r, thus found for a particular value of R, does actually enable us to calculate from the known electromotive force for open circuit, with a moderate degree of approximation in the case of a constant battery, and also, but less surely, in the case of a secondary battery, what available difference of potential

will exist between the terminals of the battery when connected by other and somewhat widely differing values of R, and therefore also to find what arrangement of a battery it will be best to adopt in any given circumstances. So far as this practical result is concerned, the numerous methods which have been devised for the determination of the resistance of a battery before any sensible polarization (which requires time to develop) has been set up are, though interesting in themselves, of no practical value, and we shall not here describe any of them.*

From equation (31) we have

$$r = \frac{E - V}{V} R. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

To determine r therefore we have simply to measure with a potential galvanometer the difference of potential which exists between the terminals of the battery when on open circuit, or connected only by the galvanometer coil, the resistance of which we suppose to be very great in comparison with r, and again to measure in the same way the difference of potential when the terminals are connected by a resistance R, also small in comparison with that of the galvanometer.†

If the galvanometer scale be graduated so that readings are proportional to the tangents of the corresponding angles, we have, if D be the deflection in the first case, and D' the deflection in the second case, the equation

$$r = \frac{D - D'}{D'} R. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

* Some account of the Measurement of Electrolytic Resistance will be found in the author's larger treatise, vol. i.

† If the battery consist of a large number of cells, it may be divided into sections and so tested, or each cell may have its resistance measured separately.

Instead of a potential galvanometer a quadrant electrometer may be employed if the battery is not too large, and the same formula applies when D and D' are taken proportional to the sines of the angles through which the mirror is turned.

A resistance coil, which may be of German silver wire, constructed as described in p. 232, should be used for the resistance connecting the terminals, and if the current passing through it be considerable its resistance should be determined when the current is flowing. This may be done by including in its circuit a current-galvanometer, and determining the current γ through the wire in amperes, when V is read off in volts on the potential instrument. The resistance of the wire with that of the current-galvanometer is in ohms V/γ, and this is to be used as the value of R in equation (33).

If a galvanometer of high resistance be not available, an approximate test can be made by means of a sensitive galvanometer of low resistance. The battery and galvanometer are joined in series with a resistance R, and again with a resistance R'. Let D and D' be the deflections, which must have a difference comparable with either. Then, supposing E and r to be the same in both cases, and putting G for the resistance of the galvanometer we have

$$D = m\frac{E}{R + G + r}, \quad D' = m\frac{E}{R' + G + r},$$

where m is a constant.

Therefore we find

$$r = \frac{D'R' - DR}{D - D'} - G \quad . \quad . \quad . \quad . \quad (34)$$

Mance has shown how to determine the resistance of

a battery by means of Wheatstone's Bridge. The battery is placed in the position *BD* of Fig. 52 above, and a key is connected between *A* and *B*. The resistances r_1, r_2, r_3 are adjusted until the depression of the key produces no alteration in the galvanometer deflection. The galvanometer and the key, with their respective connecting wires, are then conjugate conductors;* and it is easy to show that the resistance of the battery is then r_2r_3/r_1. The needle of the galvanometer is kept nearly at zero by means of a small magnet during the adjustment of the resistances, so that it is as sensitive as possible to any alteration of current produced by depressing the key.

This method is so troublesome as to be practically useless, chiefly on account of the variation of the effective electromotive force of the cell produced by altération of the current through the cell which takes place when the key is depressed. Prof. O. J. Lodge † has discussed the method, and shown how it may be improved by inserting a condenser in series with the galvanometer between *C* and *D*. Still it is inconvenient and gives no information which may not be obtained more easily in another way, and we shall therefore not enter into further detail regarding it.

Sir William Thomson‡ has however shown how the same mode of operating may be made to give the resistance of a galvanometer when there is no other galvanometer available. The arrangement of Fig. 52 is varied by placing the galvanometer in the position *BD*, and a key in the position there shown as

* That is an electromotive force in either produces no current in the other. See the author's larger treatise, vol. i., p. 159.

† *Phil. Mag.* 1877, p. 515.

‡ *Proc. R.S.* Vol. xix. (Jan. 1871).

occupied by the galvanometer. The deflection of the galvanometer produced by depressing the battery key is nearly annulled by means of a magnet, and the resistances r_1, r_2, r_3 are adjusted until no alteration of the galvanometer deflection takes place when the key in *CD* is depressed. When this is the case *C* and *D* are at the same potential, since the addition of the conductor *CD* does not disturb the current distribution in the network; and we have for the resistance r_4 of the galvanometer

$$r_4 = \frac{r_2}{r_1} r_3.$$

CHAPTER IX.

THE MEASUREMENT OF ENERGY IN ELECTRIC CIRCUITS.

WHEN a circuit in which a current of electricity is flowing contains a motor, or machine by which work is done in virtue of electromagnetic action, the whole electrical work done in the circuit consists, as was first shown by Joule, of two parts, work spent in heat in the generator and motor and in the conductors connecting them, and work done in moving the motor against external resistance. The total rate at which electrical energy is given out in the circuit is, as we have seen, $E\,C$ watts, where E is the total electromotive force of the generator in volts, and C is the number of amperes of current flowing. The rate at which work is spent in heat is in watts, by Joule's law, C^2R, where R is the total resistance in circuit in ohms; hence, if we call W the rate at which work is done in the motor,* we have,

$$EC = C^2R + W \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We may write this equation in the form,

$$C = \frac{E - \dfrac{W}{C}}{R} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* We consider here a system in which C is constant, and neglect loss of energy due to local currents, &c., in the motor. For fuller information regarding motors and their action see a paper by Profs. Ayrton and Perry, *Proc. Soc. Tel. Engs.*, 1883, republished in the electrical journals, also Prof. S. P. Thompson's *Dynamo-Electric Machinery.*

which shows that the current flowing is equal to that which would flow in the circuit if, the resistance remaining the same, the motor were held at rest, and the electromotive force diminished by an amount equal to W/C. This is what is called the *back electromotive force* of the motor, and is due to the action of the motor in setting up when driven an electromotive force tending to send a current through the circuit in the opposite direction to that of the current by which the motor is driven. We shall denote the back electromotive force by E_1. Hence equation (2) becomes,

$$C = \frac{E - E_1}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the rate at which work is spent in driving the motor is $E_1 C$.

To determine E we have simply to measure with a potential galvanometer or voltmeter, the difference of potential between the two terminals of the generator. Calling this V, and R_1 the effective resistance of the generator, we have plainly,

$$E = V + CR_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, since C and also the total resistance R in the circuit can be found by measurement, we find by (3)

$$E_1 = E - CR \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where all the quantities on the right-hand side are known.

The ratio of $E_1 C$, the electrical energy spent per unit of time in the circuit otherwise than in heating the conductors, to the whole electrical energy EC spent in the circuit per unit of time, that is the ratio of E_1 to E, we may call the electrical efficiency of the arrangement.

Denoting this efficiency by e, we find, by equation (4),

$$e = \frac{E_1}{E} = 1 - \frac{CR}{E} = 1 - \frac{E - E_1}{E} \quad . \quad . \quad (6)$$

Hence the smaller C is made, that is, the slower the energy is given out, the value of the efficiency of the arrangement is the more nearly equal to *unity*, the value of the efficiency of an arrangement in which the energy in the motor done against external resistance is exactly equal to the whole electrical energy given out in the circuit.

When however energy is spent at the maximum rate in working the motor, $E_1 C$ has its greatest value. But by (5)

$$E_1 C = EC - C^2 R = W.$$

This equation may be written,

$$C^2 R - EC + W = 0,$$

a quadratic equation of which the solution is,

$$C = \frac{E \pm \sqrt{E^2 - 4RW}}{2R}.$$

Now in order that these values of C may be *real*, $4RW$ cannot be greater than E^2. Hence the greatest value, W can have is $E^2/4R$. When W has this maximum value, C is equal to $E/2R$, and therefore E_1 equal to $E/2$. Hence the electrical efficiency is $\frac{1}{2}$. It is to be very carefully observed that although in this case the arrangement is that of *greatest electrical activity*, it is *not that of greatest electrical efficiency*, as it has only about one-half the efficiency of one in which energy is given out at a very slow rate. The case is exactly analogous to that referred to in p. 85, of a battery arranged so as to give maximum current through a given external resistance.

All that has been stated above is applicable to the case of a motor fed by any kind of generator whatever. The generator employed however is generally some form of dynamo- or magneto-electric machine driven by an external motor, such as a steam- or gas-engine or a water-wheel, and a few of the results obtained below apply only to such cases, which will be indicated as they occur.

When the generator and motor are exactly similar machines, and the same current passes through both, the ratio of E_1 to E will be that of $n\,A\,f(C)$ to $n'\,A\,f(C)$; where n and n' are the speeds of the machines, A a constant depending on the form and disposition of the magnets, and $f(C)$ a function of the current. Hence in this case the efficiency is measured simply by the ratio of the speed of the motor to that of the dynamo. The more nearly therefore the speed of the motor approaches to that of the generator, the greater is the efficiency. It is to be observed however that two machines identically alike will not in practice be perfectly similar in their action, even with the same currents flowing in their armatures and field-magnet coils. The armature currents tend to weaken the field in the generator, and to strengthen the field in the motor.

In general, the higher the speed at which the motor is run, the greater is the electrical efficiency of any arrangement, for it is obvious that the higher the speed the more nearly does E_1 approach to E, and therefore the value of E_1/E, the measure of efficiency, to unity.

For a constant difference $E - E_1$, the ratio of the energy spent in heating the conductors by the current to the whole energy expended in the circuit, may be reduced by increasing the total electromotive force E of the circuit. The energy spent in heat is C^2R, or $(E - E_1)^2/R$, and the

ratio of this to $E\,C$ is CR/E. But $C\,R$ is equal to the constant difference $E - E_1$, hence the ratio is $(E - E_1)/E$, and this becomes smaller as E is increased. A greater efficiency is therefore obtained by using high potentials than by using low potentials. Hence a greater electrical efficiency is realized, with a given magneto- or dynamo-electric machine used as generator and a given motor, when both generator and motor are run at higher speeds. Consequently the generator should be run as fast as possible, and the motor loaded lightly, or the speed with which the working resistance is overcome reduced by gearing between it and the motor.

When high potentials are obtained by the use of machines wound with fine wire, or by using as generator a battery of a large number of cells joined in series to drive a high potential motor, the gain of electromotive force is accompanied by an increase of resistance in the circuit. But if we suppose the speed of the motor to be so regulated that the difference between the total electromotive force in the circuit and the back electromotive force of the motor remains the same in the different cases, it is easy to show that the electrical efficiency of the arrangement is greater for high electromotive forces than for low. If, as supposed, $E - E_1$ remains constant, while E is changed to $n\,E$, we have for the total activity of the motor $nEC - (E - E_1)\,C$. Dividing this by nEC we get for the electrical efficiency,

$$e = \frac{n-1}{n} + \frac{1}{n}\frac{E_1}{E} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As n is made greater and greater, the first term on the right becomes more and more nearly equal to unity, and the last term to zero. Hence, on the supposition made,

the efficiency is increased by increasing the working electromotive forces. Taking as a particular case $n=2$, we see that the efficiency is $\frac{1}{2}$ together with one-half of the former efficiency; if $n = 4$, the efficiency is $\frac{3}{4}$ together with one-fourth of the former efficiency, and so on for other values of n. This result holds for any case whatever in which the condition that $E-E_1$ should remain constant is fulfilled; and hence it is independent of any change that may have been made in the resistances of the generator or motor in order to obtain the higher electromotive force nE. For example, it is plain that no sensible change in the actual rate of loss by heating of the conductors by the current will be produced by increasing the resistances of the generator and motor, if these be very small in comparison with the remainder of the resistance in circuit; as, since $E-E_1$ remains constant and the resistance is practically the same as before, the current strength will not be perceptibly altered. The ratio, however, of the activity wasted in heating to the total activity will be only $1/n$th of what it was before. In the opposite extreme case, in which the generator and motor have practically all the resistance in circuit, the current, C, $(= (E-E_1)/R)$ is diminished in the ratio in which the resistance is increased; and the actual rate of loss by heat according to Joule's law, $(E-E_1)^2/R$, is diminished in the same ratio, so that, as in the former case, its ratio to the total activity nEC is $1/n$th of what it was for the electromotive force E. We see, therefore, that here also the efficiency must be the same in both cases.

We have called E_1/E *the electrical efficiency of the arrangement*, but this is not to be confounded with the efficiency of the motor itself. The activity $E_1 C$ includes the wasted activity or rate at which work is done against

frictional resistances in the motor itself, and in the gearing which connects it with its load, as well as the useful activity or rate at which it performs useful work. Hence, although the electrical efficiency of the arrangement be very great, only a small amount comparatively of the energy given to the motor may be usefully expended, and *vice versâ;* and we define therefore the efficiency of a motor at any given speed as the ratio of the useful activity to the whole activity, taking as the latter the total rate at which electrical energy is expended in the motor; that is, $E_1C + C^2R_1$, or, which is the same, VC, where V is the difference of potential between the terminals of the motor. Accordingly, if A be the useful activity, we have for the efficiency of the motor the ratio A/VC.

To determine this ratio in any particular case the motor is run at the required speed, V is measured with a potential galvanometer, and C with a current galvanometer, and their product taken, or VC is determined with some form of electrical activity-meter, while A is determined by means of a suitable ergometer. A very convenient and accurate friction ergometer may be formed by passing a cord once completely round the pulley of the motor, and hanging a weight on the downward end, while the other is made to pull on a spiral spring fixed at its upper end and provided with an index to show its extension. The weight is adjusted so that the motor runs at the required speed, while wasting all its work in overcoming the friction of the cord, and the extension of the spring is noted, and the corresponding pull found in the same units of force as those used in estimating the downward pull due to the weight. Let the weight used in any experiment be taken in grammes, and be denoted by w, and let w' be the number of grammes required to

stretch the spring by gravity to the same amount, then the total force overcome is in dynes $(w - w')\,g$, where g is the acceleration, in centimetres per second per second, produced by gravity at the place of experiment (at London $g = 981{\cdot}71$ nearly). If n be the number of revolutions per second, and c the circumference in cms. of the pulley at the part touched by the rope, the velocity with which this force is overcome is $n\,c$, and therefore the activity in ergs per second is $n\,c\,(w - w')\,g$. If A is reckoned in watts, we have the equation,

$$A = \frac{1}{10^7} n\,c(w - w')g \quad . \quad . \quad . \quad . \quad (8)$$

If $w-w'$ be taken in pounds, and c in feet, and n be the number of revolutions per *minute*, the activity in horse-power is given by

$$A = \frac{1}{33000} n\,c\,(w - w') \quad . \quad . \quad . \quad . \quad (9)$$

and in watts approximately by

$$A = {\cdot}0226\,n\,c\,(w - w') \quad . \quad . \quad . \quad . \quad (9')$$

We have now considered cases in which electrical energy is transformed into mechanical work by means of motors working by electromagnetic action, and have seen that the whole electrical activity EC in the circuit is equal to the useful activity of the motor together with the unavailable part spent in heating the conductors in circuit, and in overcoming the frictional resistances opposing the motion of the motor. Part of the electrical energy developed by a generator may however be spent in affecting chemical decompositions in electrolytic cells placed in the circuit, as, for example, in charging a secondary battery or "accumulator." Each cell in which electrolytic action

takes place, so that the result is chemical separation at the plates of the constituents of the solution acted on, opposes a counter electromotive force to that causing the current, and the work done per second in each cell over and above that spent in heat according to Joule's law (p. 75), is equal to the product of this counter electromotive force into the strength of the current. In most cases the counter electromotive force exceeds the electromotive force required to effect the chemical decompositions, and the energy corresponding to the difference of electromotive force appears in the form of what has been called *local heat* in the electrolytic cells.

In the case of a secondary battery charged by the current from an electrical generator, which is the only case we shall here consider, the activity spent in the battery while it is being charged is equal to the product of the difference of potential existing between the terminals of the battery while the current is flowing, multiplied by the strength of the current. Let V be this difference of potential in volts, and C the current strength in amperes, then VC joules is the whole work per unit of time spent in the battery. The whole activity spent in the circuit is EC, or $VC + C^2R$, where E is the total electromotive force of the generator, and R is the resistance of the generator and the wires connecting it with the secondary. Again if E_1 volts be the electromotive force of the secondary battery, which may be measured by removing the charging battery for an instant and applying a potential galvanometer to the terminals of the secondary, the activity actually spent in charging the battery may be taken as E_1C. Hence the ratio of the activity spent in charging the battery to the whole activity in the circuit is $E_1/(V+RC)$ or E_1/E, and the

activity wasted in heating the conductors in circuit is $(E - E_1)C$. This ratio E_1/E is the same as that found above in the case of a generator and a motor, and may be called as before the electrical efficiency of the arrangement.

Hence, in order that as nearly as possible the whole electrical energy given out in the circuit may be spent in charging the battery, as many cells should be placed in circuit as suffice to nearly balance the electromotive force E of the generator, that is, the charging should be made to proceed as slowly as possible. In practice, however, a very slow rate of charging is not economical, as the work spent in driving the generator, if a dynamo- or magneto-electric machine, against frictional resistances would be greater than the useful work done in the circuit; and if the speed of the generator slackened for a little the battery would tend to discharge through it.

As in the case of the motor (p. 265), the electrical efficiency of the arrangement can be increased by increasing E and E_1, so that $E - E_1$ is maintained constant. E may, in the present case, be increased by running the generator faster, or by using a machine adapted to give higher potentials. As before, if E be increased to $n\,E$, while E_1 is changed to E'_1 so that $nE - E'_1 = E - E_1$, the electrical efficiency becomes $(n - 1)/n + E_1/nE$ as in (7) above.

The electromotive force of a Faure or storage cell is about 2·2 volts when fully charged, but is considerably less when nearly discharged. When the cell is placed in the charging circuit, the counter electromotive force which it opposes rises quickly to a little less than this value, and thereafter gradually increases, while the charging current falls in strength. In order to measure, therefore, the whole energy spent in charging a secondary battery, we must either use some form of integrating energy-meter which

gives accurate results, or measure, at short intervals of time, V with a potential galvanometer, and C with a current galvanometer placed permanently in the circuit. After the battery has been charged, the total number of joules spent is obtained by multiplying each value of VC by the number of seconds between the instant at which the corresponding readings were taken and that at which the next pair of readings were taken, and adding all the results. Or, more exactly, values V and C are obtained for each interval by finding the arithmetic means of the values of V and of C at the beginning and end of each interval, and taking the product of these two means as the value of the activity for that interval. Each product is multiplied by the number of seconds in the corresponding interval, and the sum of the products is the whole energy spent. The integral work in joules having been thus estimated, the efficiency of the battery may be obtained by finding in the same manner the total number of joules given out in the external working circuit while the battery is discharging. The ratio of the useful work thus obtained to the whole work spent in charging is the efficiency of the battery. In discharging in an electric light circuit, the greatest economy is obtained when the resistance of the working part of the circuit is very great in comparison with that of the battery and main conductors. Neglecting the latter part of the resistance, we see that, if a large number of lamps are arranged in multiple arc, a large number of cells should also be joined in multiple arc, so that, while the requisite difference of potential is obtained the resistance of the battery is still small in comparison with that of the external circuit.

As regards the measurement of energy spent in electric light circuits, in which continuous currents are flowing,

we have already sufficiently indicated above (Chap. VII.) how this may be done. To find the activity, or work spent per unit of time, in any part of a circuit, we have only to find the dffference of potential, V, in volts between its extremities with a potential galvanometer, and the current, C, in amperes flowing through it with a current instrument. If the activity be constant, we have simply to multiply VC by the number of seconds in any interval of time, to find the number of joules spent in that time in the part of the circuit in question. If the activity is variable, the whole energy spent in any time may be estimated by finding VC at short intervals of time, and calculating the integral as explained above (p. 272).

So far we have been considering only measurements made in the circuits of batteries or of continuous current generators. Alternating machines in which the direction of the current is reversed two or three hundred times a second are, however, frequently employed, especially in electric light circuits, and it is necessary therefore to consider the methods of electrical measurement available in such cases. We shall consider briefly, first, a class of instruments some of which can be used in a circuit of either kind, and we shall deal in the first place with their application in continuous current circuits.

These are instruments the fundamental principle of which is the mutual electromagnetic action between two circuits, to be calculated conveniently in most cases in which this can be done, by replacing each circuit according to Ampère's theory, by an equivalent magnetic shell (p. 42), and considering the mutual action of the systems. Sir William Thomson's Electric Balances, described in Chap. VII., are instruments which act on this principle, and can

be used both in alternating and in continuous current circuits.

Weber's well-known electrodynamometer* is another instrument of this class, but arranged to be used as an absolute or standard current-meter. It may be considered as constructed by replacing the needle of the standard tangent galvanometer (Chap. IV.) with a coil of radius small in comparison with that of the galvanometer coil, and suspended by a torsion wire or bifilar, so as to hang in equilibrium when no current is passing through it, with its plane at right angles to that of the large coil. Hence when a current C is made to flow through the fixed coil, and a current C' through the suspended coil, a couple is exerted on the latter, tending to set it with its plane parallel to the large coil, and this tendency is resisted by the action of the torsion or bifilar suspension, so that there is equilibrium for a deflection θ, the magnitude of which plainly depends on the product CC' of the two current strengths. To avoid disturbance from the action of the local horizontal magnetic force, the large coil may be placed parallel to the direction of that force, and the small coil brought back when deflected by the current to the initial position at right angles to the large coil, by turning the upper end of the suspension wire or wires through a measured angle.

By Ampère's theorem (p. 42) the suspended coil is equivalent to a small needle of moment nAC', where n is the number of turns of wire in the coil, A their mean area. Hence if N be the number of turns in the large coil, r its mean radius, we have as in p. 47 for the electromagnetic couple on the suspended coil, when brought back to the

* See Maxwell's *Electricity and Magnetism*, vol. ii. p. 330.

initial position, the value $2\pi/r . NC'.nAC'$ or $2\pi/r . nNACC'$. This couple is balanced by the opposite couple given by the supension, the magnitude of which for all angles within a certain range is supposed known from experiment. Calling this latter couple L, we have

$$CC' = \frac{Lr}{2\pi nNA} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the two coils be joined in series so that the same current flows through both, we have $C = C'$, and therefore

$$C^2 = \frac{Lr}{2\pi nNA} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

With such an instrument therefore an absolute measurement of a current can be made without its being necessary first to determine H.* In practical work the instruments on this principle usually employed are such as require to have their constants determined by comparison with standard instruments, such as a standard tangent galvanometer, or a standard dynamometer, and are dealt with in Chap. VII. We may here mention, however, Siemens' electro-dynamometer, in which a suspended coil is acted on by a fixed coil, and the strength of the current deduced, by means of a table of values for different angles, from the torsion which must be given to a spiral spring to bring the coil back to the zero position.

When an instrument on this principle is arranged for use as an activity-meter, one set of coils, the fixed or the movable, is made of thick wire so as to carry the whole current in the circuit, while the other set is made of high resistance and is connected to the two ends of the part of

* For fuller particulars regarding absolute instruments of this class see *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

the circuit in which the electrical activity is to be measured. In this case the force or couple required to restore the movable coils to the zero position is proportional to the product VC of the difference of potential and current, that is to the activity, for that part of the circuit; and if the instrument has been properly graduated this can be at once read off in watts, or in any other chosen units of activity. Instruments of this kind have been made by Professors Ayrton and Perry, Sir William Thomson, and Sir William Siemens. Sir William Thomson's form is described above (Chap. VII.).

We shall now consider the measurement of currents and differences of potential, and therefore also of electrical energy in the circuits of alternating machines or of transformers. In all such circuits the march of the current in each complete alternation may be stated roughly as a rise from zero to maximum in one direction, then a diminution to zero, then a change of sign and a rise to maximum in the opposite direction, followed by a diminution again to zero. The law according to which these changes take place is more or less complex in the various cases, and the complete mathematical representation of the current strength at any time would require an application of Fourier's method of representing any arbitrary periodic function, by means of an infinite series of simple harmonic terms of the form $A_i \sin (int - \epsilon_i)$, where n is 2π divided by the period T of a complete alternation, A_i and ϵ_i constants and i any integer. It has been found experimentally by M. Joubert that the variation of electromotive force in a Siemens' alternating machine can be expressed by the single harmonic term $E \sin nt$, where we reckon t from the instant at which the electromotive force was zero when changing from the direction reckoned as negative

to that reckoned as positive. There is good reason to believe that this law is not very seriously in error for the majority of alternating machines, and we shall assume its truth in what follows. The current strength is affected by the action of self-induction (p. 206) to a greater or less extent in all such machines independently of the disposition of the external circuit, especially if the revolving armature contains iron; but, as shown below, it follows, with a difference in phrase, the same law as does the electromotive force. The effect of variations in the field magnets produced by the rotating armature has also in a rigorous theory to be taken into account, but this effect, in well-designed machines without iron in their armatures is not great, and where experiments have been made to detect it, has been found to be slight, and we shall therefore neglect it.

Writing then C for the current, at a time t, less than $T/2$, reckoned from the instant at which the current was zero, we have

$$C = A \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The whole quantity of electricity generated in a half period $T/2$ is therefore

$$\int_0^{T/2} C\,dt = A \int_0^{T/2} \sin nt\,dt = \frac{AT}{\pi} \quad . \quad . \quad (13)$$

Hence if C denote the mean current in that time, we have

$$C_m = \frac{2A}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now if an electro-dynamometer be placed in the circuit so that the same current passes through both its fixed and movable coils, the current in both will be reversed at the

same instant, and their mutual action will be the same for the same current strength, and will be proportional to C^2 that is to $A^2 \sin^2 nt$. If the period of the alternation be small in comparison with the period of free oscillation of the movable coil system of the dynamometer, the mutual action of the fixed and movable coils will be the same as if a continuous current C' given by the equation

$$C'^2 = \frac{1}{T}\int_o^T C^2\,dt = \frac{A^2}{T}\int_o^T \sin^2 nt\,dt \quad . \quad . \quad (15)$$

were kept flowing through them. But by integration

$$C'^2 = \frac{A^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and substituting from (14) in this equation, we get

$$C_m = \frac{2\sqrt{2}}{\pi}C' = \cdot 9003\;C'. \quad . \quad . \quad . \quad (17)$$

In order therefore to find the actual mean current strength in the circuit of an alternating machine from the value of C' given by a current dynamometer we must multiply the latter by ·9; in other words the mean current strength is 9/10 of the strength of the continuous current which would give the same deflection. The product, if C' has been taken in amperes, multiplied by the number of seconds in any interval of time during which the machine has been working uniformly on the same circuit, will give the number of coulombs of electricity which has flowed through the circuit in that time.

The measurement of potentials is however attended with more difficulty on account of the effect of the self-induction of any electromagnetic instrument which can be applied to the circuit for this purpose. The following

method of employing Sir William Thomson's quadrant electrometer for this purpose has been used by M. Joubert.* The needle of the instrument is left uncharged, and the charging rod connected with it and used as a third electrode. If then we suppose the needle connected to one point in the circuit at which the potential is V, one pair of quadrants at a point at which the potential is V_1, and the other pair at a third point where the potential is V_2, then if D be the deflection of the spot of light corresponding to the angle (supposed small) through which the needle has been turned against the bifilar suspension, then as we have seen above (p. 134) we have

$$D = k(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad . \quad (18)$$

where k is a constant. If the needle be connected to the pair of quadrants whose potential is V_1, then we have as in (4) of Chap. VII.

$$D = \frac{k}{2}(V_1 - V_2)^2 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The constant k is determined by a process of calibration with known differences of potential similar to that described at p. 136 above.

The multicellular or the vertical voltmeter described in Chap. VII. may (preferably) be used instead of the quadrant electrometer, except when three points at different potentials are to be connected to the electrometer at the same time, as in the measurement described below, p. 300. Any doubt as to the applicability of the expression on the right of (19), with k a constant, is avoided, for in these

* *Comptes Rendus*, July, 1880. *Annales de Chimie et de Physique*, May, 1883.

instruments the values of different deflections on the scale have been fixed by experiment.

If the terminals of the electrometer employed be connected to any two points in the circuit of a machine in which the period of alternation is short in comparison with the free period of the needle, the couple acting on the needle will be at each instant proportional to the second power of the difference $V_1 - V_2$ of potential existing between these two points at that instant, and this of course is independent of the sign of the difference. Also as in the similar case of the dynamometer above, the deflection of the needle will be the same as that which would be produced by a constant difference of potential V given by the equation

$$V^2 = \frac{1}{T}\int_o^T (V_1 - V_2)^2 \, dt.$$

If we denote the actual mean difference of potential by V_m, then since the difference of potential follows the same law of variation as the current we get also

$$V_m = \cdot 9003 \, V \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If we know the resistance in the part of the external circuit between the points at which the electrometer electrodes are applied, then calling this resistance R, and supposing that this part of the circuit contains no motor or other arrangement giving a back electromotive force, and that the ratio of its self-induction to the period of alternation is zero or negligible in comparison with R, we have for the mean value of the current V_m/R, and thus by means of an electrometer alone we can measure not

only the difference of potential between the ends of, but also the current in, that portion of the circuit.

It has been pointed out by Lord Rayleigh (*Phil. Mag.* May, 1886) that the resistance offered by a conductor to the passage of a current through it is greater the smaller the period of alternation. This variation is due to the fact that as the alternation increases in rapidity the current is more and more confined by inductive action to the outer strata of the conductor which is therefore virtually reduced in section. This is not to be confounded with the fictitious increase of resistance seen in the expression $\sqrt{R^2 + n^2L^2}$ (see p. 293 below) which arises directly from the electromotive force of self-induction, but is a real increase of the value of R for the current in question. A very useful table of the resistances of conductors at different periods of alternation drawn up by Mr. W. M. Mordey and given in his paper on Alternate Current Working (*loc. cit.* p. 282) will be found in the Appendix. The theoretical data from which the table has been calculated were given by Sir William Thomson as an addendum to his presidential address (1889) to the Institution of Electrical Engineers. An abstract of it is printed with the table.

Denoting by A_m the mean value of the electrical activity in this part of the circuit, still supposing the self-induction of this part to be negligible, we have plainly

$$A_m = \frac{1}{RT}\int_0^T (V_1 - V_2)^2\,dt = \frac{V^2}{R} \quad . \quad . \quad (21)$$

In the same way, since the value of the electrical activity at any instant is C^2R, we have from the results of experiments made by an electro-dynamometer,

$$A_m = \frac{R}{T}\int_0^T C^2\,dt = C'^2 R \quad . \quad . \quad . \quad (22)$$

From these two results we get

$$A_m = VC' \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

that is, the mean value of the electrical activity is equal to the product of the square root of the mean square of the difference of potential, by the square root of the mean square of the current strength. It can therefore be determined by means of an electrometer and an electrodynamometer of negligible self-induction without its being necessary to know the resistance.

We shall now consider the case in which the self-induction cannot be neglected. Let R be the total resistance in the circuit, C the current flowing in it at the time t, E the total electromotive force of the machine, and L the coefficient of self-induction, or the "inductance" (as, following a usage introduced by Mr. Oliver Heaviside, we shall henceforth call it) for the whole circuit, that is, the number which multiplied into dC/dt gives the electromotive force opposing the increase or diminution of the current. We shall suppose L a constant, although there can be no doubt that in some alternating machines its value is different in different positions of the armature. The iron cores of the field magnets act to a greater or less extent as cores for the armature coils, and as the magnetic susceptibility of iron is a function of the strength of the magnetizing current, L, which is the magnetic induction through the armature produced per unit of its own current, must vary accordingly.

Still for certain alternators which have no iron in their armatures the variation of L with the position of the armature is slight.* It will also be assumed that there

* See the discussion on Mr. W. M. Mordey's paper on "Alternating Current Working" Inst. of Elect. Eng. May, 1889 (*The Electrician*, May 24, 31, June 7, Aug. 2, 1889).

are no masses of metal in which local currents can be generated moving in the field. On these assumptions the equation of the current is

$$RC = E - L\frac{dC}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

But by the law which we have assumed for the machine,

$$E = ne \sin nt = E_0 \sin nt. \quad . \quad . \quad . \quad . \quad (25)$$

where e is a constant such that E_0 is the maximum value of E for the given speed. Substituting in (24) we get

$$L\frac{dC}{dt} + RC = E_0 \sin nt \quad . \quad . \quad . \quad . \quad (26)$$

which integrated becomes

$$C = Ae^{-\frac{R}{L}t} + \frac{E_0}{\sqrt{R^2 + n^2L^2}} \sin(nt - \epsilon) \, . \quad (27)$$

where

$$\sin \epsilon = \frac{nL}{\sqrt{R^2 + n^2L^2}}, \quad \cos \epsilon = \frac{R}{\sqrt{R^2 + n^2L^2}} \, . \quad (28)$$

The term $Ae^{-\frac{R}{L}t}$ is only important immediately after the circuit is closed, and will therefore be neglected.

We may remark that if L were equal to zero (27) would reduce to $C = E_0/R \, . \sin nt$, which corresponds to (12) above.

From (27) we get for the mean current

$$C_m = \frac{2E_0}{T(R^2+n^2L^2)^{\frac{1}{2}}}\int_{\epsilon/n}^{\epsilon/n+T/2} \sin(nt - \epsilon)\,dt = \frac{2E_0}{\pi(R^2+n^2L^2)^{\frac{1}{2}}} \, (29)$$

Also for the mean square of the current strength as

given directly by an electro-dynamometer we have by (27) the equation

$$C'^2 = \frac{E_0^2}{T(R^2 + n^2 L^2)} \int_0^T \sin^2(nt - \epsilon)\, dt$$

$$= \frac{1}{2} \frac{E_0^2}{R^2 + n^2 L^2} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and we have therefore as before, the relation

$$C_m = \cdot 9003\ C'.$$

From (27) we see that the effect of self-induction is to diminish every value of the current in the ratio of $E_0/(R^2 + n^2 L^2)^{\frac{1}{2}}$ to E_0/R, and to produce a retardation of phase which measured in time is ϵ/n seconds; that is, the resistance is virtually increased in the ratio $(R^2+n^2L^2)^{\frac{1}{2}}/R$, and the current in following the law of sines passes through any value ϵ/n seconds after it would have passed through the corresponding value if there had been no self-induction. It is plain also that, for any finite resistance R, by diminishing T, that is, by increasing the speed of the machine, the current can, by (25), be made to approach the limiting value

$$C = \frac{e}{L} \sin\left(nt - \frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad (31)$$

which is independent of the resistance, and has a retardation of phase of $T/4$ seconds, a quarter period of a complete alternation. Hence integrating over a half period from zero current to zero current again, and dividing by $T/2$ we get for the maximum mean current

$$C_m = \frac{2e}{\pi L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

To find the mean value A_m of the total electrical activity in the circuit, we have by (25), and (27)

$$A_m = \frac{1}{T}\int_o^T E\,C\,dt = \frac{E_0^2}{(R^2 + n^2L^2)^{\frac{1}{2}}}\int_o^T \sin(nt - \epsilon)\sin nt\,dt$$

$$= \frac{1}{2}\frac{E_0^2 R}{R^2 + n^2L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence by (30)

$$A_m = C'^2 R \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

that is, the true mean value of the total electrical activity is equal to the mean square of the current strength multiplied by the total resistance in circuit.

It may be shown, from (33), by the method used in p. 85 above that the total activity in the circuit is greatest when $R = nL$, that is, *for a given speed and a given value of* L, the activity is a maximum when $R = nL$. It must be observed however that *for a given resistance* R the activity is greater the smaller the value of T, that is, the greater the speed. When R has the value nL we have, by (27) $\epsilon = \pi/4$; that is, the retardation of phase is then one-eighth of the whole period.*

If in the circuit there be two sources of electromotive force of the same period T but of different phases; for example, two machines driven so as to have the same period of alternation, the solution here given applies. For the two electromotive forces combine to give a single electromotive force of the same period as the components

* The conclusions as to maximum work and retardation of phase, as well as most of the theoretical results stated above as to the action of alternating machines, were first we believe given by M. Joubert, *Comptes Rendus*, 1880. The problems of alternating machines joined in series, or in parallel, or as motors, were considered by Dr. J. Hopkinson in a lecture to the Inst. of Civil Engs. 1883, and in a paper "On the Theory of Alternating Currents," Soc. Tel. Engs. and Els., Nov. 1884. The principal results of this latter paper are reproduced below. See also Mr. Mordey's paper *loc. cit.*

but differing in phase from either; so that, to use the solution it is only necessary to take this resultant electromotive force as $E_0 \sin nt$, reckoning the time from an instant at which $\sin nt$ is zero and increasing. If the difference of phases be 2ϕ reckoned in angle, the interval between the successive instants at which a component is increasing through zero is $2\phi/n$. Hence taking the zero of reckoning of time midway between these two instants we may denote the two components by $E_1 \sin (nt + \phi)$, $E_2 \sin (nt - \phi)$. Calling their resultant $E_0 \sin (nt - \psi)$, we have

$$E_0 \sin (nt - \psi) = E_1 \sin (nt + \phi) + E_2 \sin (nt - \phi) \quad (35)$$

By elementary trigonometry we get

$$\left. \begin{aligned} E_0^2 &= E_1^2 + E_2^2 + 2 E_1 E_2 \cos 2\phi \\ \text{and} \qquad \tan \psi &= \frac{E_2 - E_1}{E_1 + E_2} \tan \phi \end{aligned} \right\} \quad . \quad . \quad (36)$$

When $\phi = 0$, $\psi = 0$, and $E_0 = E_1 + E_2$, as is evident without calculation, since the machines are then in the same phase. If $E_1 = E_2$, that is if the machines are equal, the resultant is in phase halfway between its components. When this is the case we have also

$$E_0 = 2 E_1 \cos \phi \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which when $\phi = 0$ gives, as it ought, $E_0 = 2E_1$.

Considering still two unequal machines and remembering that when the value of the resultant electromotive force is increasing through zero, the value of the current is given by (27), that then the electromotive force of the leading machine is $E_1 \sin (nt + \phi + \psi)$, and that of the following machine $E_2 \sin (nt - \phi + \psi)$, we have for the mean activity A_{1m} of the leading machine.

$$A_{1m} = \frac{1}{T}\int_o^T E\,C\,dt$$

$$= \frac{E_0 E_1}{T(R^2 + n^2 L^2)^{\frac{1}{2}}}\int_o^T \sin(nt - \epsilon)\sin(nt + \phi + \psi)\,dt$$

$$= \frac{1}{2}\frac{E_0 E_1}{(R^2 + n^2 L^2)^{\frac{1}{2}}}\cos(\phi + \psi + \epsilon)$$

$$= \frac{1}{2}\frac{E_0 E_1}{R^2 + n^2 L^2}\left\{R\cos(\phi + \psi) - nL\sin(\phi + \psi)\right\} \quad (38)$$

To find the mean activity of the following machine we have only to change the sign of ϕ in this expression. We get

$$A_{2m} = \frac{1}{2}\frac{E_0 E_2}{R^2 + n^2 L^2}\left\{R\cos(\phi - \psi) + nL\sin(\phi - \psi)\right\} \quad (39)$$

If the machines be equal $E_1 = E_2$, and $\psi = 0$, so that

$$A_{1m} = \frac{E_1^2 \cos\phi}{R^2 + n^2 L^2}(R\cos\phi - nL\sin\phi)\ . \quad (40)$$

$$A_{2m} = \frac{E_1^2 \cos\phi}{R^2 + n^2 L^2}(R\cos\phi + nL\sin\phi)\ . \quad (41)$$

Since ϕ is less than $\pi/2$, both $\cos\phi$ and $\sin\phi$ are positive, and therefore the following machine does more work than the leading machine. Hence, unless each is completely controlled by the prime-mover, the leading machine will increase its lead, and this will go on until $2\phi = \pi$, when the two machines will be in exactly opposite phases, and will exactly neutralize one another. This tendency to assume opposition of phase depends on the difference $A_{2m} - A_{1m}$, and this having the factor $nL/(R^2 + n^2L^2)$, has a maximum value, for a given resistance and a given period of alternation, when $nL = R$.

The machines thus arrange themselves so that no current passes in the wires joining their terminals, and these wires alternate in relative potential with the period of the machines, and each is at any instant very approximately at one potential throughout. It might therefore be inferred that if a working circuit be joined from one wire to the other, a current will pass through that circuit, and that the two machines will control one another so as to keep in the same phase in supplying it. We shall consider this case as a further example of the theory.

Let 2ϕ be the difference of phase with reference to the external circuit, so that at time t, $E \sin (nt+\phi)$, $E \sin (nt-\phi)$ are the electromotive forces of the two machines, C_1, C_2 the currents, L the coefficient (supposed constant) of self-induction for each, r the resistance of each machine from one point of attachment to the other point, and R the resistance of the external circuit. We shall suppose that the external circuit has no sensible self-induction, and that the whole work there developed is spent in overcoming resistance, for example, in lighting glow lamps. By considering the circuit through each machine and the external resistance,* remembering that the current in the latter is $C_1 + C_2$, and therefore the difference of potential between the terminals $R(C_1 + C_2)$, we find the equation

$$\left.\begin{aligned} L\frac{dC_1}{dt} + rC_1 + R(C_1 + C_2) &= E \sin (nt + \phi) \\ L\frac{dC_2}{dt} + rC_2 + R(C_1 + C_2) &= E \sin (nt - \phi) \end{aligned}\right\} \quad (42)$$

* According to Kirchhoff's rule, p. 88 above, taking into account the electromotive force of self-induction in each circuit.

Adding and subtracting we get

$$L\frac{d}{dt}(C_1 + C_2) + (2R + r)(C_1 + C_2)$$

$$= 2E\cos\phi \,.\, \sin nt \quad . \quad . \quad . \quad . \quad (43)$$

$$L\frac{d}{dt}(C_1 - C_2) + r(C_1 - C_2) = 2E\sin\phi \,.\, \cos nt \quad (44)$$

Solving these we find as in (27)

$$C_1 + C_2 = \frac{2E\cos\phi}{\{(2R + r)^2 + n^2L^2\}^{\frac{1}{2}}}\sin(nt - \epsilon) \quad (45)$$

$$C_1 - C_2 = \frac{2E\sin\phi}{(r^2 + n^2L^2)^{\frac{1}{2}}} \,.\, \cos(nt - \epsilon') \quad . \quad (46)$$

where

$$\tan\epsilon = \frac{nL}{2R + r}, \quad \tan\epsilon' = \frac{nL}{r} \,. \quad . \quad . \quad (47)$$

Hence if A_{1m} be the mean activity of the leading machine

$$A_{1m} = \frac{E}{2T}\int_o^T (C_1 + C_2 + C_1 - C_2)\sin(nt + \phi)\,dt$$

$$= \frac{E^2}{T}\left\{\frac{\cos\phi}{\{(2R+r)^2 + n^2L^2\}^{\frac{1}{2}}}\int_o^T \sin(nt - \epsilon)\sin(nt + \phi)dt\right.$$

$$\left. + \frac{\sin\phi}{(r^2 + n^2L^2)^{\frac{1}{2}}}\int_o^T \cos(nt - \epsilon')\sin(nt + \phi)dt\right\}$$

$$= \frac{1}{2}\frac{E^2}{(2R+r)^2 + n^2L^2}\{(2R+r)\cos^2\phi - nL\sin\phi\cos\phi\}$$

$$+ \frac{1}{2}\frac{E^2}{r^2 + n^2L^2}(r\sin^2\phi + nL\sin\phi\cos\phi) \quad . \quad . \quad (48)$$

The mean activity A_{2m} of the following machine may be got from A_{1m} by altering the sign of ϕ throughout the expression on the right. Hence

$$A_{2m} = \frac{1}{2}\frac{E^2}{(2R+r)^2+n^2L^2}\{(2R+r)\cos^2\phi+nL\sin\phi\cos\phi\}$$
$$+\frac{1}{2}\frac{E^2}{r^2+n^2L^2}(r\sin^2\phi - nL\sin\phi\cos\phi) \quad . \quad . \quad (49)$$

$A_{1m} - A_{2m}$ is positive, that is more work is done by the leading than by the following machine. The lead will therefore tend to zero, and the machines to settle down into coincidence of phase with reference to the external circuit, that is, into opposite phases with reference to their own circuit, which agrees with the result already obtained.

We shall consider only one more case of this theory, that of an alternating motor connected by its terminals to two conductors upon which an alternating difference of potential is impressed by other machines. Let the motor be started so as to have the same period of alternation. Then denoting by R the resistance of the motor-armature and the leads up to the point at which the difference of potential is impressed, by L the co-efficient of self-induction for the same part of the circuit by $E_1 \sin(nt+\phi)$, the impressed difference of potential at time t, by $E_2 \sin(nt-\phi)$, the back electromotive force of the motor at the same instant, we have for the equation of the current

$$L\frac{dC}{dt} + RC = E_1\sin(nt+\phi) - E_2\sin(nt-\phi) \quad (50)$$

This equation differs only in the sign of E_2 from that from which (38) and (39) above are deduced. Hence

taking the value of A_{2m} in (41) we have for the mean electric activity received by the motor

$$A_{2m} = \frac{1}{2}\frac{E_0 E_2}{R^2 + n^2 L^2}\{R\cos(\phi - \psi) + nL\sin(\phi - \psi)\} \quad (51)$$

where

$$\left.\begin{aligned} E_0 &= (E_1^2 + E_2^2 - 2E_1E_2\cos 2\phi)^{\frac{1}{2}} \\ \tan\psi &= -\frac{E_1 + E_2}{E_1 - E_2}\tan\phi \end{aligned}\right\} . \quad (52)$$

The second equation of (52) gives

$$\cos\psi = (E_1 - E_2)\cos\phi / E_0, \quad \sin\psi = -(E_1 + E_2)\sin\phi / E_0,$$

and these values substituted in (51) yield

$$A_{2m} = \frac{1}{2}\frac{E_2}{R^2 + n^2 L^2}\{E_1(R\cos 2\phi + nL\sin 2\phi) - E_2R\} \quad (53)$$

Now 2ϕ being the difference of phase, cannot be numerically greater than π, and therefore the work *received* by the motor is less when 2ϕ is negative than when it is positive, that is, is less when the motor is leading than when it is following. Hence the motor will tend to run slower when leading and faster when following; or the difference of phase will tend towards zero. Also as long as 2ϕ is not far from zero A_{2m} is less the greater the lead, and greater the greater the lag, and in nearly the same proportion. Hence when the machines are once in phase any small deviation is opposed by a proportional corrective tendency. This depends almost entirely on the term involving the factor $nL/(R^2 + n^2L^2)$ in the value of A_{2m} given in (53), and therefore for a given resistance R, and period of alternation T, has its greatest value when $nL = R$, or, $L/R = T/2\pi$.

Writing in (53)

$$\sin 2\phi' = \frac{R}{(R^2 + n^2 L^2)^{\frac{1}{2}}}, \quad \cos 2\phi' = \frac{nL}{(R^2 + n^2 L^2)^{\frac{1}{2}}} \tag{54}$$

we get

$$A_{2m} = \frac{1}{2} \frac{E_2}{R^2 + n^2 L^2} \left\{ E_1 (R^2 + n^2 L^2)^{\frac{1}{2}} \sin 2(\phi + \phi') - E_2 R \right\} \tag{55}$$

which is obviously a maximum when $\phi + \phi' = \pi/4$. We have then

$$A_{2m} = \frac{1}{2} \frac{E_2}{R^2 + n^2 L^2} \left\{ E_1 (R^2 + n^2 L^2)^{\frac{1}{2}} - E_2 R \right\} \tag{56}$$

This value of A_{2m} is positive if

$$\frac{E_1}{E_2} > \frac{R}{(R^2 + n^2 L^2)^{\frac{1}{2}}}$$

which may be the case even if $E_2 > E_1$. Hence we have the curious result that an alternating machine may act as a motor even if its electromotive force be greater than the impressed or driving electromotive force.

The theory just given of the working of alternating machines on the same circuit is due to Dr. J. Hopkinson, F.R.S. (Soc. Tel. Engs. and Els., Nov. 1884). Its conclusions were verified by him in 1884 in experiments made with two De Meriten's machines made for the lighthouse at Tino. Some very striking experiments are described in Mr. Mordey's paper referred to above (p. 281), and it contains moreover much interesting practical information on this subject. Difference of opinion at present exists as to whether Mr. Mordey's results are in accordance with the mathematical theory. It is to be remembered however that the theory does not take account of the action of the armature currents of the field magnets nor of the variation of self-induction. The whole subject requires further investigation.

We may apply (as we have already done in the problem of the alternating motor just treated) the formulas given above for the whole circuit to a part, taking for E the impressed electromotive force on the part of the circuit considered, and for R and L the proper values for that part only. We find that the effect of self-induction on the current is virtually to increase the resistance from R to $\sqrt{R^2+n^2L^2}$, and to produce a difference of phase between the current and E given also by (27) and (28). But the resistance of a conductor is (p. 75) the activity spent in it by unit current in producing heat; hence by (22) and (34) the resistance in this sense is not increased. The quantity $\sqrt{R^2+n^2L^2}$, or one analogous if the simple harmonic law is not followed, is very important, and Mr. Oliver Heaviside has proposed for it the very appropriate name *impedance.**

The impedance of a current electro-dynamometer or current balance, through both coil-systems of which flows the whole current in the main circuit, cannot if it be low (as it generally is) in comparison with that of the rest of the circuit, affect appreciably the strength of the current by its introduction; and since the whole current passes through both sets of coils, the instrument will give the mean square of the current passing.

It may be otherwise, however, with a fine wire instrument used as a shunt to measure the difference of potential between two points of the circuit. The inductance of such an instrument may be considerable, and if it be used alone its impedance will seriously affect the result.

* The Electrical Congress held at Paris in August last (1889) voted in favour of calling this quantity the "résistance apparente" of the part of the circuit considered. The term *impedance* has however been approved by the British Association Committee on Electrical Standards in its Report to the Newcastle meeting held in Sept. last (1889). For its definition see Note in Appendix.

Since the value of the impedance depends on the period of alternation, it will have different values when connected to circuits in which the periods are different. To obviate the uncertainty and inconvenience arising from this cause, the instrument is made sensitive enough to allow a considerable non-inductive resistance to be joined in series with its own coils. This makes the value of $R/\sqrt{R^2 + n^2 L^2}$ approximately unity. Some calculations made by Prof. T. Gray * for Sir William Thomson's vertical scale voltmeter, give for this ratio with only the resistance of the instrument (640 ohms) included, and a period of alternation of $\frac{1}{100}$ of a second, the value ·9976, which is within $\frac{1}{4}$ per cent. of unity. Plainly the error caused by the impedance in this case is small with any period commonly employed, and can be made still smaller by the introduction of non-inductive resistance. The difference of phase between the currents through the coils of the instrument, and the difference of potential (given by (28) above) is therefore small. This difference of phase, it is to be remembered, does not affect the value of the mean square of the difference of potential, provided the amplitude be corrected for the effect of inductance.

It is however of importance in the action of a wattmeter, of which one coil is placed in the main circuit, and the other as a shunt between the extremities of the portion of the circuit in which the activity is to be estimated. For let the circuit divide into two parts, each forming a derived circuit on the other, and L_1, L_2, R_1, R_2, C_1, C_2 be the inductances, the resistances, and the maximum currents in the two parts, C the maxi-

* *Electrical Review*, Jan. 20, 1888. The instrument here referred to is not described above, but is similar in principle and construction to the instrument shown on p. 99. It has however only two fixed and two movable coils and reads with a pointer on a vertical scale.

mum total current in the circuit, and ϵ_1, ϵ_2 the differences of phase between C and C_1, C_2 respectively, a similar analysis to that given above shows that if there be no mutual induction between the two parts of the circuit

$$\left.\begin{aligned} \frac{C_1^2}{R_2^2+n^2L_2^2} = \frac{C_2^2}{R_1^2+n^2L_1^2} &= \frac{C^2}{(R_1+R_2)^2+n^2(L_1+L_2)^2} \\ \tan\epsilon_1 &= \frac{n\,(L_1 R_2 - L_2 R_1)}{(R_1+R_2)R_2 + n^2\,(L_1+L_2)\,L_2} \end{aligned}\right\} \quad (57)$$

with a similar formula for ϵ_2.

If either L_1, L_2 be both small, or $L_1/L_2 = R_1/R_2$, the difference of phase between the two currents C_1, C_2 will be insensible. If the first condition is fulfilled both parts of the circuit will have currents agreeing in phase with the difference of potential between the terminals, and, on the usually allowable supposition of negligible mutual induction, a wattmeter whose coils are included in them will measure accurately the power expended. It will, on the same supposition, also measure accurately the power expended *while the wattmeter is on circuit* if the ratio $R/\sqrt{R^2 + n^2 L^2}$ be approximately unity for the fine wire circuit, since the main current passes through the other coil, and it can be shown that the deflection will be the same as would be produced by a constant activity A_m given by the equation

$$A_m = \frac{1}{T}\int_0^T VC\,dt \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Where V, C, are the values of the difference of potential and the current at time t. If also $\sqrt{R^2 + n^2 L^2}$ for the thick wire coil be small in comparison with the same quantity for the part of the main circuit in which the activity is being measured, the inclusion of the wattmeter

will not affect the circuit and the activity shown by the instrument may be taken as that existing when it is not applied.

But it is important to notice that the second condition may be fulfilled without an accurate measurement of power by the wattmeter. The currents through the two coils will have the same phase, but this may not be the phase of the difference of potential between the terminals if between these there be sensible self-induction in the main circuit; and the wattmeter would give too high as result. It is essential for accuracy that the current through the fine wire coil-system of the wattmeter should have the phase of the difference of potential, while that in the thick wire coil should have the phase of the main current.

The general problem of finding the ratio of the apparent activity as shown by the wattmeter to the true activity can be solved with great ease by aid of the theory given above. For let A, B be the points at which the terminals of the fine wire coil-system are attached to the main circuit; let R_1, R_2, L_1, L_2 be the resistances and inductances of the fine wire and thick wire circuits between A, B, and C_1, C_2 the currents in them, then by (27) if the difference of potential between the terminals A, B is $E_0 \sin nt$.

$$\left.\begin{aligned} C_1 &= \frac{E_0}{(R_1^2 + n^2 L_1^2)^{\frac{1}{2}}} \sin (nt - \epsilon_1) \\ C_2 &= \frac{E_0}{(R_2^2 - n^2 L_2^2)^{\frac{1}{2}}} \sin (nt - \epsilon_2) \\ \text{where} \quad & \\ \tan \epsilon_1 &= \frac{nL_1}{R_1}, \quad \tan \epsilon_2 = \frac{nL_2}{R_2} \end{aligned}\right\} \quad . \quad . \quad (59)$$

The current through the fine wire is therefore the same as if the resistance in its circuit between the points A, B

were without inductance, and the difference of potential E between A, B had the value

$$E_0 R_1/(R_1^2 + n^2 L_1^2)^{\frac{1}{2}} \,.\, \sin(nt - \epsilon_1).$$

Hence if $A'm$ be the apparent activity in the main circuit between A, B

$$A'm = \frac{1}{T}\int_0^T E C_2 dt$$

$$= \frac{1}{T}\frac{E_0^2 R_1}{(R_1^2+n^2L_1)^{\frac{1}{2}}(R_2^2+n^2L_2^2)^{\frac{1}{2}}}\int_0^T \sin(nt-\epsilon_1)\sin(nt-\epsilon_2)dt$$

$$= \frac{1}{2}\frac{E_0^2 R_1 \cos(\epsilon_1 - \epsilon_2)}{(R_1^2 + n^2L_1^2)^{\frac{1}{2}}(R_2^2 + n^2L_2^2)^{\frac{1}{2}}} \quad \text{[by (38)]}$$

$$= \frac{1}{2}\frac{E_0^2 R_1 (R_1 R_2 + L_1 L_2)}{(R_1^2 + n^2 L_1^2)(R_2^2 + n^2 L_2^2)} \quad . \quad . \quad (60)$$

since by (59)

$$\sin\epsilon_1 = nL_1/(R_1^2 + n^2 L_1^2)^{\frac{1}{2}}, \ \cos\epsilon_1 = R_1/(R_1^2 + n^2 L_1^2)^{\frac{1}{2}}$$

with similar values for $\sin\epsilon_2$, $\cos\epsilon_2$.

But if Am be the true mean activity then in the same way

$$Am = \frac{1}{T}\frac{E_0^2 \cos\epsilon^2}{(R_2^2 + n^2 L_2^2)^{\frac{1}{2}}}\int_0^T \sin nt \sin(nt - \epsilon_2)\, dt$$

$$= \frac{1}{2}\frac{E_0^2 \cos\epsilon}{(R_2^2 + n^2 L_2^2)^{\frac{1}{2}}} = \frac{1}{2}\frac{E_0^2 R_2}{R_2^2 + n^2 L_2^2} \quad . \quad (61)$$

Hence

$$\frac{Am}{A'm} = \frac{R_2}{R_1}\frac{R_1^2 + n^2 L_1^2}{R_1 R_2 + L_1 L_2} = \frac{1 + n^2\tau_1^2}{1 + n^2\tau_1\tau_2} \quad . \quad (62)$$

where τ_1, τ_2 are written for L_1/R_1, L_2/R_2 respectively, the so-called "time constants" of the two parts of the circuit.

Now in general $\tau_1 < \tau_2$, hence as a rule the wattmeter will give too high a result. If the time constants and the period of alternation be known, then A_m can be calculated from the apparent activity by this equation.*

In any case in which a wattmeter is inapplicable, if the actual resistance of the portion of the circuit considered is known, and the mean square of the current can be measured with accuracy, the product of the two will, as shown above (p. 285), be the true mean value of the activity. This of course will be given in watts, if the resistance is taken in ohms and the current in amperes.

As we have seen above, the proper mean value of the current, and of the difference of potential, and therefore also of the activity, can be found for any part of a circuit in the case of negligible self-induction, either by means of an electro-dynamometer, or by means of an electrometer, when the resistance of the part of the circuit is known. When the resistance is unknown or uncertain, as for example in the case of incandescence lamps, the current and difference of potential may be measured for the lamp circuit in the following manner. A coil of german silver wire, having a resistance considerably greater than that of the lamps as arranged, constructed as described above (p. 188), so as to have no self-induction, is connected in series with a current-meter between the terminals of the machine so as to be a shunt on the lamps. The lamps are brought to their normal brilliancy, and the mean square C'^2 of the current through the german silver wire measured. If R be the resistance of this wire, including,

* This result was given without proof by Prof. Ayrton in his remarks on "Testing the Efficiency of Transformers," *Proc. Soc. Tel. Eng. and El.* Feb. 1888.

For methods of determining time-constants, see the author's *Theory and Practice*, &c., vol. ii.

if appreciable, the resistances of the current-meter and its connections, and R be great in comparison with the coefficient of self-induction of the current-meter divided by T, we have for the mean square V^2 of the difference of potential between the terminals of the lamp system, the value C'^2R^2. The current-meter is now employed to measure the whole current flowing to the lamps while their brilliancy is kept the same. Denoting the mean square of this current by $C_1'^2$, we have for the value A_m of the mean activity spent in the lamp system, the equation

$$A_m = VC' = C'C_1'R \quad . \quad . \quad . \quad . \quad (63)$$

An electrometer may be used in the following manner to give the mean square of the current, and of the difference of potential for any part of a circuit whether containing motors, or arc lamps, or any arrangement with or without counter-electromotive force or self-induction. A coil of thick german silver wire (or to prevent sensible heating a set of two or more equal coils arranged in multiple arc) having no self-induction is included in the part of the circuit considered, so that the current to be measured also flows through the wire. The mean square of the difference of potential between the ends of this resistance is measured as described above (p. 279) by connecting one pair of quadrants of the electrometer to one end, and the needle and the other pair of quadrants to the other end, and the mean square C''^2 of the current found by dividing by the square of the resistance of the wire. The mean square of the difference of potential between the terminals of the part of the circuit considered, is then found in the same manner. The product is not generally to be taken as the mean square of the activity in the part of the circuit

considered, for it is evident that in this case what is obtained is the value of

$$\frac{1}{T}\int_0^T V^2 dt \times \int C^2 dt,$$

where V and C are the difference of potential and the current at any instant. The square root of this quantity is not generally the same thing as

$$\frac{1}{T}\int_0^T VC\,dt$$

the true mean value of the activity. This is, however, given directly by the following method.*

Let the two ends of the resistance coil of zero self-induction and known resistance R be called A and B, and let the extremities of the portion of the circuit for which the measurements are to be made be called C and D. One of the pairs of quadrants is connected to A, the other pair to B, and the needle to C, and the reading d say taken. The quadrants remaining as they were, the needle is connected to D, and the reading d' taken. Now if at any instant V_1 be the potential of A, V_2 of B, V_1' of C, and V_2' of D, we get by (18) above

$$\left.\begin{aligned} d &= \frac{k}{T}\int_0^T (V_1 - V_2)\left(V_1' - \frac{V_1 + V_2}{2}\right)dt \\ d' &= \frac{k}{T}\int_0^T (V_1 - V_2)\left(V_2' - \frac{V_1 + V_2}{2}\right)dt \end{aligned}\right\}. \qquad (64)$$

* This method is described by A. Potier, *Journal de Physique*, t. ix. p. 227, 1881, but was independently invented also by Prof. W. E. Ayrton, and Prof. G. F. Fitzgerald (see Prof. Ayrton on "Testing the Power and Efficiency of Transformers," *Proc. Soc. Tel. Engs. and Els.*, Feb. 1888).

and by subtraction and division by kR

$$\frac{d - d'}{kR} = \frac{1}{TR}\int_0^T (V_1 - V_2)(V_1' - V_2')\,dt\,. \quad (65)$$

But we have seen that the expression on the right hand side of (65) is the true mean value of the activity required.

If $V_1' - V_1$ be great in comparison with $V_1 - V_2$ and A, say, be connected with the case of the instrument, the first of (64) becomes

$$d = \frac{k}{T}\int_0^T (V_1 - V_2)\,V_1'\,dt \quad . \; . \; . \quad (66)$$

If A and D coincide $V_2' = V_1$, and the activity in the part of the circuit between C and D is, by (61), given by (62) alone when put in the form

$$\frac{d}{kR} = \frac{1}{TR}\int_0^T (V_1 - V_2)\,V_1'\,dt \quad . \; . \; . \quad (67)$$

This observation is due to Mr. Sayers, a pupil of Professor Ayrton. It is thus possible in the case supposed to use an electrometer as a direct reading wattmeter.

If a quadrant electrometer is used as here explained, care must be taken to see that the equation (17) holds for the instrument, and if it does not hold what the deviation from fulfilment of this relation is. Dr. Hopkinson found (*Phil. Mag. Ap.* 1885), that the indications of his instrument were very exactly expressed by the equation

$$D = \frac{k}{1 + mV^2}(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right)$$

where m is a small constant. Hence for high values of V it was necessary to know and use this second constant.

The deviation from fulfilment of the ordinary equation here shown was found to be in great part due to downward electrical force on the needle caused by its hanging a little too low in the quadrants.

If the needle hangs at its proper level and is otherwise properly adjusted, and the quadrants are close, the equation may be taken as accurate enough for practical purposes, but as a precaution it should be tested by a battery which can be made to give different relatively known differences of potential.

CHAPTER X.

MEASUREMENT OF INTENSE MAGNETIC FIELDS.

WE have seen above (p. 45) that every element of a conductor carrying a current in a magnetic field is acted on by a force tending to move it in a direction at right angles to its length and to the direction of the resultant magnetic force at the element, and have stated how the magnitude of the force may be calculated in terms of the intensity of the field and the strength of the current. Hence if we know the strength of the current flowing in a conductor placed in a magnetic field, and measure the force exerted in virtue of electromagnetic action on any element of the conductor, we can calculate the intensity of the field at the element. On this principle are founded the following simple methods, mainly suggested by Sir William Thomson, of determining in absolute measure the intensity of magnetic fields in dynamo machines or other electromagnetic apparatus.

We shall take first the case of two long straight pole faces oppositely magnetized and placed at a short distance apart, facing one another, with their lengths vertical. In the middle of the space between the poles a wire, *w* (Fig. 65), somewhat longer than the poles, so as to extend a little above and below them, is hung vertically, by a cord

of four or five feet in length, attached near its upper end from a fixed peg above, and is stretched by the weight, W, attached near its lower end. Two pendulums, made

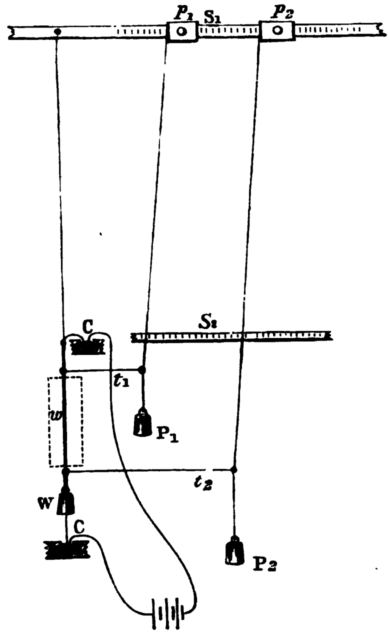

FIG. 65.

of weights, P_1, P_2 carried by fine threads, are hung from two sliding pieces which can be moved along a graduated cross-bar S_1 above, so placed that the ends are as nearly as possible in a plane parallel to the pole faces

and passing through the middle of the space between them. The two pendulum threads and the wire, w, are thus nearly in one plane. One of these pendulums is made so long as to have its bob below the level of the lowest part of the pole faces, while the other has its bob a little below the level of the top of the pole faces, and the former is placed at the greater distance from the suspended wire. A thin thread attached to the upper end of the suspended wire is carried out horizontally and made fast at its other end to the suspension thread of the nearer pendulum.

A similar thread is attached at one end to a point of the wire near the bottom of the pole faces, and carried out similarly and made fast at the other end to a point nearly on the same level in the suspension thread of the further pendulum. The upper and lower ends of the wire, w, are placed, as shown, in mercury cups, to which are also connected the electrodes of a battery, by means of which a current can be sent through the wire w, and measured by means of a galvanometer in the circuit. A scale, S_2, is placed a little behind the plane of the threads, so that the position of a point in each, on the same level near their lower ends, can be easily read off.

When an experiment is made, the sliding pieces, p_1 p_2, are moved towards the left until the threads, t_1 t_2, are quite slack, and the positions of each thread on the upper and lower scales are read off and noted. The position of the wire, w, when t_1, t_2 are quite slack is also marked at the upper and lower ends of the pole faces or elsewhere. A current is then sent through the wire, w, in such a direction that the electromagnetic force acting on it moves it towards the left. The sliding pieces, p_1, p_2 are then moved towards the right so as to cause the pendulums to pull the wire by means of the threads, t_1, t_2, back again to its initial

position. When the upper and lower ends have come back to their former positions, the electromagnetic force on the wire is balanced by the pulls exerted by the pendulums. The positions of the pendulum threads are again read off on the upper and lower scales and noted with the strength of the current flowing in w. From these results we can easily calculate the average intensity of the field at the place occupied by the wire, w. For let W be the mass of each of the pendulum bobs in grammes, d the distance through which the top of the pendulum thread has been carried by p_1 to the right of the point of the thread opposite to the lower scale S_2, d_2 the corresponding distance for the other pendulum, l the vertical distance between the levels of the tops of the pendulum threads and the lower scale measured in the same units as d_1, d_2, L the length of the opposed pole faces, and C the strength of the current in C.G.S. units (one-tenth the number of amperes). The downward force in dynes on each of the masses in Wg, where g is the acceleration in centimetres per second per second (=981·4 in latitude of Glasgow) produced by gravity in a falling body at the place of experiment. The total pull to the right exerted by the threads on the wire is therefore $Wg(d_1 + d_2)/l$, and this is equal to the pull towards the left on the wire produced by the electromagnetic action. If I be the average intensity of the field along the wire in C.G.S. units, we have for this pull in dynes $I\,L\,C$. Hence we get the equation

$$I\,L\,C = Wg\frac{d_1 + d_2}{l},$$

and therefore,

$$I = \frac{Wg}{LC} \cdot \frac{d_1 + d_2}{l} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In an experiment made on September 16, 1882, with a similar arrangement, W was 100 grammes, l 100 centimetres, C ·188 in C.G.S. units of current, L 30 centimetres, and $d_1 + d_2$ 25·84 centimetres. Hence,

$$I = \frac{100 \times 981{\cdot}4}{30 \times {\cdot}183} \cdot \frac{25{\cdot}84}{100} = 4496.$$

The wire, w, should not be so flexible as to bend perceptibly under the influence of the forces to which it is subjected, so that the value of I found may be nearly enough the average value of the intensity along a straight line in the space between the pole faces.

In cases in which, as in many dynamo machines, the opposite pole faces of the electromagnets are at a considerable distance apart, with or without pieces of soft iron in the intermediate space, it is practically useful to find simultaneously the magnetic field intensity along two lines in the same plane, one in the vicinity of each pole face. This may be done by so placing the electromagnets that the two lines along which the field is measured are in a horizontal plane, and using, instead of the single wire carrying the current, a rectangle of copper wire, or strip, of which the opposite sides are in these lines, supported on knife-edges in the bisecting line parallel to the pole face so that it can turn round that line as axis. The frame should be weighted symmetrically on the two sides of the line of knife-edges, so that it rests with just enough of stability in the horizontal position. The ends of the wire or strip forming the rectangle are brought out one above the other at one of the knife-edges with a piece of insulating material between them, and bent over so that the end of each dips into a mercury-cup in line with the knife-edges. The electrodes of a battery are connected

to the mercury-cups and a measured current is sent round the rectangle. Since the poles have opposite magnetisms, the electromagnetic action causes one side of the rectangle to move upwards, the other side to move downwards, and thus turns the rectangle round the knife edges.

The moment of the electromagnetic forces is balanced by the action of weights, which may be riders of known weight made of wire, placed on the sides of the rectangle, which is thus brought back to its initial position. If we call I the average intensity of the fields along the two sides of the rectangle in the equilibrium position, and C the current strength, both as before measured in C.G.S. units, L the length of each side, and d the distance between them in centimetres, the moment of the electromagnetic forces round the knife-edges is $I\,C\,L\,d$. The opposite moment resisting the motion is, if only one weight of W grammes at a distance of d' cms. from the line of knife-edges is used, $W g d'$. Hence, equating these moments, we get

$$I = \frac{W g}{C L} \frac{d'}{d}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which I can be calculated. If more than one weight, W, is used, each must be multiplied by its distance from the line of knife-edges, and the sum of the products multiplied by g for the equilibrating moment.

In some cases it may be convenient to use more than one turn of wire in the rectangle. If there be n turns, each of length L, $n\,L$ is to be used instead of L in the formula above.

An obvious modification of this arrangement, which may be useful in some cases, is a rectangle suspended in a vertical plane, and kept in equilibrium in the proper

position when no current is flowing through it, by means of a bifilar suspension, or a single thread or thin wire under torsion. When a current is sent through the frame, it is deflected round a vertical axis by the electromagnetic action, and is brought back to the initial position of equilibrium by means of two pendulums, the points of suspension of which are on sliding pieces which can be moved along horizontal parallel bars fixed above at right angles to the plane of the rectangle when in the equilibrium position, and in the same vertical planes as its sides. Each pendulum cord has attached to it a thread which pulls horizontally at the middle of one side of the rectangle. When the rectangle is deflected, the sliding pieces are moved in opposite directions, so that, in consequence of the opposite inclinations of the pendulums to the vertical, forces restoring equilibrium are applied to the rectangle. As before, we have for the electromagnetic couple $ICLd$. Supposing the two points of suspension of the pendulums to be on one level, and the points of attachment of the pulling threads to the pendulum cords to be on a level lower by a distance of l cms., the distances of the verticals through the points of suspension from the corresponding verticals through the attachments of the threads to the pendulum cords to be d_1, d_2 cms. for the respective pendulums, and W grammes the mass of each bob, we have, for the moment of the equilibrating forces, the value

$Wg \cdot \frac{d_1 + d_2}{l} d.$ Hence, equating moments, we get

$$I = \frac{Wg}{CL} \cdot \frac{d_1 + d_2}{l} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If IL is the same for both sides of the rectangle, d_1 and

d_2 will be equal; but in general there will be a small difference between the two values.

In some important practical cases the pole faces are of small area and are at only a small distance apart. If there is room, a small rectangular coil, similar to that of a siphon recorder (see Fig. 66), but of comparatively few turns of wire, and without an iron core, may be hung, as described above, between the poles, with its plane parallel to the lines of force, by a bifilar or a torsion thread or wire, and a measured current sent through it. A rigid projecting arm fixed to the coil at the middle of its upper end and at right angles to the plane of the coil, has resting against it the suspension thread of a pendulum, attached at its upper end to a sliding piece movable along a horizontal bar carrying a millimetre scale, above and at right angles to the projecting arm; and by this means the coil is brought back to the initial position. When no current is flowing through the coil, the thread is allowed to hang vertically just touching the bar and the reading on the scale above noted. Let the difference between this reading and that obtained when the pendulum is deflected be d, and let l be the vertical height of the point of suspension above the projecting arm. The horizontal force exerted by the pendulum is $Wg.d/l$, and the moment of this round the vertical axis about which the coil turns $Wgr.d/l$, where r is the distance of the pendulum thread from the central plane of the coil. If n be the number of turns in the coil, b cms. its mean breadth, and L cms. the mean length of each side, the moment of the electromagnetic forces is $n\,b\,I\,L\,C$. We have, therefore,

$$I = \frac{Wgrd}{nLCbl} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This method has frequently been used for the determina-

tion of the magnetic field-intensity of the magnets of siphon recorders. The coil hanging in its place was used as the measuring coil, and when no current was flowing through it, was kept hanging vertically in stable equilibrium with its plane parallel to the lines of force by the bifilar threads attached beneath it. These threads were kept taut and bearing against the bridge B by the weights W, resting on a plane slightly inclined to the vertical. A current from one or two cells was then sent through the coil, and the difference of potential between the terminals of the coil measured by means of a potential galvanometer. The thread of the pendulum was made to pull against the projecting aluminium arm to which the siphon is attached as shown in the figure, so as to bring the coil back to the initial position. The value of d was then read off, and that of C deduced from the known resistance of the coil and the result of the measurement with the galvanometer, and being substituted with those of the other quantities W, n, b, &c. in (4), gave the value of I.

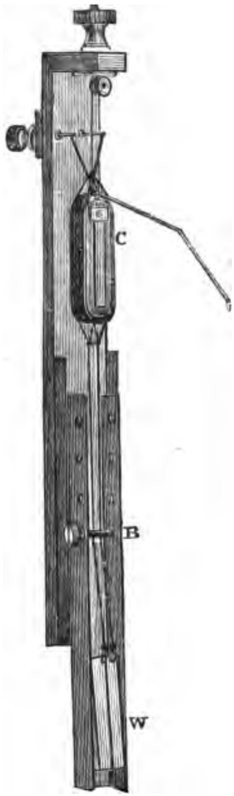

FIG. 66.

The field intensities of siphon recorders have sometimes been determined by the following method, which is interesting theoretically.

Advantage is taken of the signal-coil, which consists of a rectangular coil a little more than 5 cms. long and 2 cms. broad, made of thin wire and supported by a silk thread above, so as to hang in a vertical plane round a rectangular core of iron, which nearly fills, but nowhere touches, the coil. To the lower end of the coil two silk threads are attached, as shown in Fig. 66, and are stretched against a bridge *B* by two weights resting on the inclined plane *W*. This bifilar arrangement gives a directive force, tending to bring the plane of the coil into parallelism with that of the bifilar threads; so that when the coil is disturbed from that position, which is one of stable equilibrium, and then left to itself, it will, if the circuit be not closed, vibrate about the position of equilibrium with a determinate period of oscillation, with slowly diminishing range, until at last it comes to rest. But if the circuit be closed through a high resistance, the coil will come more rapidly to rest; and if we gradually diminish this resistance, deflecting the coil through the same angle and noting its subsidence at each diminution, we shall find it come more and more quickly to rest, until a resistance is obtained with which in circuit it just returns to the position of equilibrium without passing that position. When this resistance has been determined, the strength of the field can be calculated.

Let θ be the deflection of the coil from the position of equilibrium at time t, and T its period of oscillation when the circuit is not closed. We have then, neglecting the resistance of the air and other disturbances, for the equation of motion,

$$\frac{d^2\theta}{dt^2} + \frac{4\pi^2}{T^2}\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let now the circuit of the coil be closed; a retarding force due partly to air-resistance, but in the main to the current induced in the wire, and, if the effect of self-induction be neglected, proportional to the angular velocity, will act on the coil; and the equation of motion or this case will be of the form

$$\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + \frac{4\pi^2}{T^2}\theta = 0 \quad . \quad . \quad . \quad . \quad (6)$$

For let I be the mean intensity of the magnetic field over the space occupied by the coil at time t, L the inductance of the circuit for that position of the coil, R the total resistance in the circuit, μ the moment of inertia of the coil round a vertical axis passing through its centre, l the effective length of wire in the coil (that is, the length of wire in its two vertical sides), and b the mean half-breadth of the coil. If we call N the number of lines of force which pass through the coil at time t, and γ the strength of the induced current in the coil at that instant, we have plainly

$$N = bIl \sin\theta - L\gamma.$$

The rate at which N increases per unit of time is therefore

$$\frac{dN}{dt} = bIl\cos\theta\frac{d\theta}{dt} + bl\sin\theta\frac{dI}{dt} - \frac{d}{dt}(L\gamma);$$

and if θ be small, and I be therefore supposed the same for every position of the coil, we have approximately

$$\frac{dN}{dt} = bIl\frac{d\theta}{dt} - \frac{d}{dt}(L\gamma).$$

But dN/dt is the electromotive force due to the inductive

action; hence the current γ is by Ohm's law given by the equation

$$\gamma = \frac{bIl}{R}\frac{d\theta}{dt} - \frac{1}{R}\frac{d}{dt}(L\gamma).$$

It was assumed that the second term of this expression for γ would prove negligible in comparison with the first; and this assumption was so far justified by the results of the experiments, which agreed fairly well with results obtained, for other instruments of the same pattern, by a modification of the second method described below.

The couple due to the action of the field on the current is $bIl\gamma$; and therefore, on the supposition of negligible self-induction, the retardation of the angular velocity of the coil at time t is

$$k\frac{d\theta}{dt} = \frac{b^2 I^2 l^2}{\mu R}\frac{d\theta}{dt}.$$

Hence (2) becomes

$$\frac{d^2\theta}{dt^2} + \frac{b^2 I^2 l^2}{\mu R}\frac{d\theta}{dt} + \frac{4\pi^2}{T^2}\theta = 0 \quad . \quad . \quad . \quad (7)$$

The motion represented by this differential equation will be oscillatory or non-oscillatory, according as the roots of the auxiliary quadratic are imaginary or real—that is, according as $4\pi/T >$ or $< b^2 I^2 l^2/\mu R$. Hence, if R be the critical resistance at which the motion just ceases to be oscillatory, we have

$$I^2 = \frac{4\pi R\mu}{Tb^2 l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When l and b are expressed in centimetres, μ in grammes and centimetres, T in seconds, and R in cms. per second, I is given by this equation in absolute C.G.S. units of magnetic field intensity.

The method of experimenting consisted in first finding the value of T, the free period of vibration of the coil with its circuit uncompleted, then finding the resistance which, being placed in circuit with the coil, just brought the needle to rest without oscillation. This resistance was conveniently obtained by means of a resistance-box included in the circuit, and therefore added no self-induction to that in the coil. An aluminium arm attached to the coil, and carrying the siphon, served as an index to render the motions of the coil visible. The resistance R was first made much too great, so as to give a slow subsidence, then gradually diminished until the value which just prevented oscillation was reached; and it was found that this value could be determined easily within 50 ohms, and with great care, to 20 ohms. As the experiments on the recorders had to be made somewhat hurriedly, and on account of the disturbances neglected, and, further, as μ was taken as equal to Wb^2, where W is the mass of the coil, the results could not be taken as giving more than a rough approximation to I: but those for two instruments are given below in illustration of the method. For both instruments the values of W, L, and b were the same, and were respectively taken as 3·343 grammes, 3338 cms., and ·95 cm. Each coil had a mean vertical length of 5·3 cm., a mean breadth of 1·9 cm., and contained 45·72 metres of fine wire arranged in 290 turns, and had a resistance of about 500 ohms.

	T.	R.	I.
(1)	·465 sec.	3330×10^9 cms. per sec.	5150 C. G. S.
(2)	·500 „	3530×10^9 „ „	5120 „

This method is obviously applicable in any case in which a coil can be suspended by a torsion wire, or bifilar,

or other arrangement so as to have a measurable free period of vibration.*

The following method, which has been frequently used in the Physical Laboratory of the University of Glasgow, is very convenient and useful in many cases. It consists in exploring the magnetic field by means of the induced current in a wire moved quickly across the lines of force over a definite area in the field. The wire is in circuit with a reflecting "ballistic" galvanometer—that is, a galvanometer the system of needles of which has so great a moment of inertia that the whole induced current due to the motion of the wire has passed through the coil before the needle has been sensibly deflected. The deflection thus obtained is noted, and compared with the deflection obtained when, with the same or a smaller resistance in circuit, a portion of the conductor is made to sweep across the lines of force over a definite area of a uniform field of known intensity, such as that of the earth or its horizontal or vertical component.

In performing the experiments, it is necessary to take precautions to prevent any action except that between the definite area of the field selected and the wire cutting its lines of force. For this purpose the conducting-wire, which is covered with insulating material, is bent so as to form three sides of a rectangle, the middle one of which is of the length of the portion of field to be swept over. This side is placed along one side of the space over which it is about to be moved so that the connecting wires lie along the ends of the space; and the open rectangle is then moved in the direction of its two sides

* The method just described gives (theoretically) a means of determining the ohm. For suppose the coil hung in a sufficiently intense and uniform field, the intensity of which has been measured by another method, and the decrement of the oscillatory motion produced by the induction observed. Then the resistance could be calculated.

until the opposite side of the space is reached. The connecting wires thus do not cut the lines of force, and the induced current is wholly due to the closed end of the rectangle.

Instead of a single wire cutting the lines of force, a coil of proper dimensions (for many purposes conveniently of rectangular shape), the mean area of which is exactly known, may be suspended in the field with its plane parallel to the lines of force, and turned quickly round through a measured angle of convenient amount not exceeding 90°; or it may be suspended with its plane at right angles to the lines of force and turned through an angle of 180°. If n be the number of turns, A their mean area, and I the mean intensity of the field over the area swept over in each case, then, in the first case, if θ be the angle turned through, the area swept over is $n\ A \sin\theta$ and the number of lines cut is $n\ I\ A \sin\theta$; in the second, the area is $2\ n . A$, and the number of lines cut is $2\ n\ I\ A$.

In order that with the feeble intensity of the earth's field a sufficiently great deflection for comparison may be obtained, it is necessary that a relatively large area of the field should be swept over by the conductor. One convenient way is to mount on trunnions a coil of moderately fine wire of a considerable number of turns wound round a ring of large radius, like the coil of a standard tangent galvanometer, and arranged with stops so that it can be turned quickly round a horizontal axis through an exact half-turn, from a position in which its plane is exactly at right angles to the dip. This coil, if the ballistic galvanometer is sensitive enough, may always remain in the circuit. The change in the number of lines of force passing through the coil in the same direction relatively to the coil, produced by the half-turn, is plainly equal to twice as many

times the area of the turn of mean area as there are turns in the coil (the effective area swept over) multiplied by the total intensity of the earth's magnetic force at the place of experiment. Or, and preferably when the horizontal component of the earth's magnetic force has been determined by experiment, the coil may be placed in an east and west (magnetic) vertical plane, and turned through an exact half-turn. The magnetic field intensity by which the effective area is to be multiplied is in this case the value of H.*

A sufficiently large area of the earth's field for comparison may, in some cases, be obtained very readily by carrying the wire along a rod of wood, say two or three metres long, and suspending this rod in a horizontal position by the continuations of the conductor at its ends from two fixed supports in a horizontal line at a distance apart equal to the length of the rod, and securing the remaining wires in circuit so that they may not cause disturbance by their accidental motion. The rod will thus be free to swing like a pendulum by the two suspending wires. The pendulum thus made is slowly deflected from the vertical until it rests against stops arranged to limit its motion. When the needle is at zero, the rod is quickly thrown to the other side against similar stops there, and caught. The straight conductor thus sweeps over an area of the vertical component of the earth's field equal to the product of the length of the rod into the horizontal distance between the two positions of the conductor at the extremities of its swing. The rod may be placed at any azimuth, as the suspending portions of the conductor in

* The method of reducing results of observations to absolute measure by means of an earth inductor was used by Professor H. A. Rowland in his experiments on the magnetic permeability of iron, steel, and nickel.—*Phil. Mag.*, vol. 46, 1873.

circuit, moving in vertical planes, can cut only the horizontal lines of force; and the induced currents thus produced have opposite directions and neutralize one another.

The calculation of the results is very simple. By the theory of the ballistic galvanometer* (the same *mutatis mutandis* as that of the ballistic pendulum), if q be the whole quantity of electricity which passes through the circuit, and if θ be the angle through which the needle has been deflected, or the "throw," we have, neglecting air resistance, &c.,

$$q = \frac{2}{G}\sqrt{\frac{\mu H}{m}}\sin\frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where μ is the moment of inertia of the needle and attachments, m the magnetic moment of the needle, H the earth's horizontal magnetic force, and G the constant of the galvanometer. If θ be small, as it generally has been in these experiments, we have

$$q = \frac{1}{G}\sqrt{\frac{\mu H}{m}} \cdot \theta \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the quantities of electricity produced by sweeping over two areas, A and A', are directly as the deflections.

Let A be the total area swept over in the field or portion of field the mean intensity I of which is being measured, A' and I' the same quantities for the known field, R, R' the respective total resistances in circuit, q, q' the quantities of electricity generated in the two cases, θ,

* For further particulars see the author's *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

θ' the corresponding deflections supposed both small; we have

$$q = \frac{A\,I}{R} = \frac{1}{G}\sqrt{\frac{\mu\,H}{m}}\,\theta,$$

$$q' = \frac{A'\,I'}{R'} = \frac{1}{G}\sqrt{\frac{\mu\,H}{m}}\,\theta'$$

and therefore

$$I = \frac{A'\,R\,\theta}{A\,R'\,\theta'}\,I' \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If convenient, θ and θ' may be taken as proportional to the number of divisions of the scale traversed by the spot of light in the two cases.

The error caused by neglecting the effect of air resistance, &c., in diminishing the deflection will be nearly eliminated if R and R' be chosen so that θ and θ' are nearly equal.

The following new method of reducing ballistic observations to absolute measure has been given by Sir William Thomson. A short induction coil wound round the centre of an ordinary magnetizing helix, whose length is great compared with its diameter, is kept in circuit with the galvanometer. A measured current is sent through the wire of the helix, and when the needle is at rest the circuit of the helix is broken, and the galvanometer deflection read off. If N be the number of turns of wire per cm. on the helix, C the current in electromagnetic C.G.S. units, the magnetic force within it is $4\,\pi\,n\,C$ parallel to the axis; and if A' be the proper mean area of the cross-section of the helix, and n' the number of turns in the induction coil, the number of lines (unit tubes) of force passing out of the galvanometer circuit when the current is stopped is $4\pi\,N\,n'\,A'\,C$.*

* See the author's *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol ii.

Hence, R' denoting the total resistance in circuit, the total quantity q' of electricity generated is $4\pi N n' A' C/R'$ and instead of (9) we get

$$I = 4\pi N n' C \frac{A' R \theta}{A R' \theta'} \quad . \quad . \quad . \quad . \quad (12)$$

The ballistic method of investigation was also used by Prof. H. A. Rowland (*Phil. Mag.* vol. i., 1875) for the determination of the ideal surface distribution of magnetism on magnets. A thin ring of wire was made just large enough to pass round the magnet experimented on, and was placed in circuit with a ballistic galvanometer. It was then, while encircling the magnet and held with its plane at right angles to the axis of the magnet, slided quickly along the magnet through equal short distances, and the deflection of the needle noted for each motion. The deflections thus obtained gave for thin magnets an approximate comparative estimate of the density at different points along the magnet of the surface distribution of ideal magnetic matter by which the action of the magnet could be produced, and the results were reduced to absolute measure by means of an earth-inductor.

This method, used along with Sir William Thomson's method of reduction to absolute measure, gives a very ready means of estimating with much exactness the total quantity of imaginary magnetic matter in one pole or one end of a magnet, whether of bar, horse-shoe, or other shape. The ring, which for the present purpose may be larger, and thick enough to contain any convenient number of turns, is placed at the centre or nearly neutral region of the magnet, and then quickly pulled off and away from the magnet, and the galvanometer deflection (θ) noted. A measured current is then sent through the helix, and the deflection (θ') produced by suddenly opening the circuit of

the helix also observed. Let n be the number of turns in the ring of wire, and ϕ the total quantity in C.G.S. units of imaginary magnetic matter on the portion of the magnet swept over, then the number of lines of force cut through by each turn of wire in the ring is $4\pi\phi$, and if R be the total resistance in circuit, the total quantity (q) of electricity generated is $4\pi n\phi/R$. We have therefore

$$q = \frac{4\pi n\phi}{R} = \frac{2}{G}\sqrt{\frac{\mu H}{m}}\sin\frac{\theta}{2},$$

and for the helix we get from the calculation above

$$q' = \frac{4\pi N n' A' C}{R'} = \frac{2}{G}\sqrt{\frac{\mu H}{m}}\sin\frac{\theta'}{2}.$$

By division we get

$$\phi = NA'C\frac{n' R\sin\frac{\theta}{2}}{n R'\sin\frac{\theta}{2}} \quad . \quad . \quad . \quad . \quad (13)$$

and if the deflections are small angles,

$$\phi = NA'C\frac{n' R\theta}{n R'\theta'} \quad . \quad . \quad . \quad . \quad (14)$$

This equation is of course also applicable to the reduction to absolute measure of the results of determinations of magnetic distribution made by the ballistic method. The value of ϕ deduced for each deflection divided by the area of the corresponding small portion of the magnet is approximately the surface density of the ideal distribution, the distribution on the end faces being of course included in the end deflections.

CHAPTER XI.

THEORY OF THE DIMENSIONS OF THE UNITS OF PHYSICAL QUANTITIES.

WE have, in p. 70 above, explained the term *change-ratio* of a physical quantity: in order to show clearly the relations of the various absolute units of electrical and magnetic measurement to the units on which they are based, we shall here investigate for each of the principal quantities the formula of dimensions from which the numerical value of the change ratio is to be found in any particular case.

A physical quantity is expressed numerically in terms of some convenient magnitude of the same kind taken as unit and compared with it. The expression of the quantity consists essentially of two factors, a *numeric*,* and the *unit* with which the quantity measured is compared; and

* The term *numeric* has been introduced by Prof. James Thomson (Thomson's "Arithmetic," Ed. LXXII., p. 4). It denotes a number, or a proper fraction, or an improper fraction, or an incommensurable ratio. We shall find it convenient to employ it here where we wish to lay stress on the fact that we are dealing with what are essentially numerical expressions. Of course what is actually meant by the convenien ly brief expressions "a length, L," "a mass, M," "a force, F," and the like, is simply that L, M, F, &c., denote the numerics which express the respective quantities in terms of the units chosen, that is, are, as we shall say below, the *numerics for the quantities* in terms of those units. Further in such phrases as "the product of mass and velocity," or "the product of charge and potential," and so on, the product (or whatever other function is specified) of the numerics is of course what is intended. If all such expressions were made verbally unexceptionable, the resulting prolixity would be intolerable.

the numeric is the ratio of the quantity measured to the quantity chosen as unit. Thus when a certain distance is said to be 25 yards, what is meant is that the distance has by some process been compared with the length, under specified conditions, of a certain standard rod (which length is defined as a yard) and the ratio of the former to the latter found to be 25.

The unit of measurement is of course itself capable of being expressed numerically in terms of any unit of the same kind, and in the same way therefore its full expression consists of a numeric and the new unit. Hence if N be the numeric for any physical quantity in terms of any unit, N' in terms of another unit, and n the numeric for the first unit in terms of the second, we have

$$N' = n \, . \, N \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In order therefore to find the numerical expression N' of the quantity in terms of the second unit from its numerical expression N in terms of the first we have to multiply by n, the ratio of the first unit to the second. This numeric has been appropriately called the *change-ratio* for the change from the first unit to the second.

The change from N to N' cannot be made unless the change-ratio n, is known. Each unit may have been arbitrarily chosen without reference to any other unit, and n determined by some process of measurement; or the units may have been derived from certain chosen fundamental units, and the ratio deduced from the relation of one system of fundamental units to the other. In the measurements described in this work the units employed are entirely of the second kind here referred to.

The task before us is to determine the manner in which the various derived units involve the fundamental units, that is, we have to determine for each quantity (p. 180 above) the change-ratio n in terms of the fundamental units. The formula which expresses n for a unit of measurement of any quantity we shall call the *Formula of Dimensions* or the *Dimensional Formula* of the quantity. To prevent the necessity for the constant repetition of these terms we shall denote the dimensional formula of any quantity, of which the numerical expression in terms of some chosen unit is denoted by any particular symbol (see p. 326, footnote), by the same symbol inclosed in square brackets. Thus we denote the dimensional formula of the quantity Q by the symbol $[Q]$.

Examples of the values of $[Q]$ will be found in dealing with the various units, to which we now proceed. We shall first consider the definitions and relations of the fundamental units in common use and the derivation from them of the units of other physical quantities. In doing so we shall find the dimensional formula in each case and its numerical values for certain changes of units.

For brevity we shall sometimes in what follows call the numerical part of the complete expression of any quantity a *numerical quantity of that kind*, as this will at once serve to indicate that we are dealing with a numeric, and refer to the quantity with which it is connected. Thus we shall frequently use the phrases, *a numerical length*, *a numerical velocity*, and the like to denote the numerics for the quantities in terms of the respective units, whatever these may be, chosen for their expression. In this way L, which denotes the numeric for a length in terms

of some chosen unit, will be called a *numerical* length, and so on for other quantities.*

FUNDAMENTAL UNITS.

(1) *Length.* The standard unit of length in Great Britain is defined by Act of Parliament in the following terms: † "The straight line or distance between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the Office of the Exchequer shall be the genuine standard of length at 62° F., and if lost it shall be replaced by means of its copies."

Authorized copies are preserved at the Royal Mint, the Royal Society of London, the Royal Observatory at Greenwich, and the New Palace of Westminster. The comparison of the length of the standard with the lengths of its copies has been effected with the utmost scientific accuracy, and formed a most elaborate and important scientific investigation.

The length of a simple pendulum which beats seconds has been determined for several places by means of very careful observations, and repeated pendulum experiments at these places would in the event of the destruction of the standard and all its copies give a means of accurately renewing them.

* In the former edition of this work L was called "the numeric of a length," and so for other quantities; but it has been pointed out to me by Prof. James Thomson that as we may say for example that 25 yards is 25 of the unit distance of the yard, so the phrase "the numeric of a length" might be supposed to mean that length multiplied by the numeric, and thus lead to confusion. He prefers to say "the numeric of the unit" which expresses the quantity referred to. It was stated in the former edition that the term was used as an abbreviation of "numerical expression"; and therefore the phrase "the numeric of a quantity" meant the numerical expression of that quantity in terms of some chosen unit, but lest there should be any ambiguity I have adopted the phraseology in the text. That the *quantity itself*, and not merely its numerical expression in terms of some unit was meant, Prof. Thomson would indicate by the adjective *intrinsic*, as in *an intrinsic length*, *an intrinsic energy*.

† 18 and 19 Vict. c. 72, July 30, 1855.

In France and in most Continental countries the standard of length is the *Metre*. This is defined as the distance between the extremities of a certain platinum bar when the whole is at the temperature 0° of the Centigrade scale. This rod was made of platinum by Borda, and is preserved in the national archives of France. As in the case of the yard, authorized copies whose lengths have been carefully compared with the standard are preserved in various places.

The metre was constructed in accordance with a decree of the French Republic passed in 1795,* which enacted, on the recommendation of a Committee of the French Academy of Sciences, consisting of Laplace, Delambre, Borda, and others, that the unit of length should be one ten-millionth part of the distance, measured along the meridian passing through Paris, from the Equator to the North Pole. The arc of that meridian extending between Dunkirk and Barcelona was measured by Delambre and Méchain, and from their results the standard metre was realized in platinum by Borda. The metre, it is to be observed, is not now defined in relation to the earth's dimensions, and later and more accurate results of geodesy have therefore not affected the length of the metre, but are themselves expressed in terms of the length which Borda's rod has at 0° C.

In the French system the decimal mode of reckoning has been adopted for multiples and sub-multiples of all the units. Thus the metre is divided into ten equal parts each called a decimetre, the decimetre into ten parts each called a centimetre, and the centimetre into ten parts each called a millimetre. Again, a length of

* Loi du 18 germinal, an iii.

ten metres is called a decametre, of one hundred metres a hectometre, and one thousand metres a kilometre. Of these, in accordance with the prevailing practice of scientific experimenters who adopted the suggestions of the B. A. Committee, the centimetre has been very generally chosen as the unit of length for the expression of scientific results, and on it as unit of length the electric and magnetic units approved by the International Congress of Electricians held at Paris in 1882 have been founded. The reason for this choice will appear when we consider the unit of mass.

We shall denote a numerical length by L. The dimensional formula is therefore $[L]$.

For example, if we wish to find from the numeric for a length in terms of the yard as unit the numeric for the same length in terms of the metre as unit, we have $[L] = \cdot 91439$, the ratio of one yard to one metre; and this of course is equal to $36/39\cdot 3704$, or $91\cdot 439/100$, &c., the ratios of the numerics for the two units directly obtained according as the inch, or the centimetre, &c., is taken as unit of comparison. Similarly the value of $[L]$ for a change from the foot as unit to the centimetre as unit is $30\cdot 47945$.

(2) *Mass.* The legal standard of mass in Great Britain is the Imperial standard pound avoirdupois, a piece of platinum marked "P. S. 1844, 1 lb.," preserved in the Exchequer Office. In the Act of Parliament (the Act already referred to) which gives authority to the standard, it is called the "legal and genuine standard of weight;" and the Act provides that if the standard is lost or destroyed it may be replaced by means of authorized copies, which are kept in the same national repositories as the copies of the standard of length.

It is to be noted that the word "weight" used in the Act, is one which is constantly used in two senses: (1) as here, to signify the quantity of matter in a body; (2) in its proper sense, to signify the downward force of gravity on the body. It is evident that these two senses are distinct. The quantity of matter in a body is invariable; the force of gravity upon the body depends on the situation of the body, and may even be zero. At a given place the forces of gravity on different bodies are, as was proved by Newton by pendulum experiments, proportional to their masses, and thus a comparison of the weights of different bodies gives a direct comparison of their masses.

The pound has been generally used in Great Britain as the unit of mass for the expression of dynamical results, but in engineering and the arts, larger units, for example, the ton, or mass of 2240 lbs., and the hundred-weight, or mass of 112 lbs., are frequently employed.

The French standard of mass is a piece of platinum called the *Kilogramme des Archives*, made also by Borda in accordance with the decree of the Republic mentioned above. It was connected with the standard of length by being made a mass as nearly as possible equal to that contained in a cubic decimetre of distilled water at the temperature of maximum density, 4° C. The comparison was of course made by weighing, and so far as this process was concerned it was possible to obtain great accuracy, but the density of water is somewhat difficult to determine with exactness, and is still in a small degree uncertain. The relation between the standards is, however, so nearly that stated above that, for practical purposes, the error may be neglected. But on account of this uncertainty it

is important to remember that the standard is *defined* as the kilogramme made by Borda, and not as the mass of a cubic decimetre of distilled water at 4° C., which it approximately equals.

A comparison between the French and British standards of mass made by Professor W. H. Miller gave the mass of the Kilogramme des Archives as 15432·34874 grains.

The gramme, defined as 1/1000 of the mass of the Kilogramme des Archives, and approximately equal to the mass of one cubic centimetre of water at 4° C., was recommended by the B. A. Commitee as the unit mass for the expression of experimental results generally, and this choice has now been ratified by the general practice of scientific men. The convenience of the adoption of this unit of mass lies in the fact that it is approximately the mass of unit volume of the substance (water at its temperature of maximum density), usually taken as standard of comparison in the estimation of specific gravities of bodies, which therefore become in this case the same numbers as the densities of the bodies.

The multiples and sub-multiples of the gramme proceed decimally, and are distinguished by the same prefixes as those of the metre.

We shall denote a numerical mass by M, and hence its dimensional formula by $[M]$.

The value of $[M]$ for a reduction from the pound as unit to the gramme as unit is 453·593, for reduction from the grain as unit to the gramme as unit 1/15·432.

(3) *Time.* The definition of equal intervals of time belongs to dynamics and cannot here be entered on, but according to it the times in which the earth turns through equal angles about its axis are to a very high degree of

approximation equal. These intervals correspond to equal intervals of time shown by a correct clock, and if the clock just goes twenty-four hours in the time of an exact revolution of the earth about its axis (or, which is the same, the interval between two successive passages of a fixed star in the same direction across the meridian of any place), showing 0h. 0m. 0s. each time a certain point of the heavens called the First Point of Aries crosses the meridian in the same direction, it is said to show sidereal time.

Though sidereal time is used in astronomical observatories, it is more convenient in ordinary civil affairs to use solar time; but as the actual solar day, the interval between two successive transits of the sun across the meridian of any place, varies in length during the year, the standard interval is the average of such intervals, and is called a *mean solar day*. On account of the orbital motion of the earth the mean solar day is about 3m. 55·9s. longer than the sidereal day.

The mean solar second, defined as 1/86400 part of the mean solar day, is taken as the unit of time for the expression of all scientific results.

We have seen that the choice of the fundamental units is entirely arbitrary, and there is nothing in their nature which entitles them in any just sense to the term "absolute." The unit of time, which is based on the period of rotation of the earth, is, we have reason to believe, subject to a slow progressive lengthening, due to tidal retardation of the earth's rotation, and possibly also to frictional resistance of the surrounding medium; and as a matter of definition, without reference in all cases to realization, it would be easy to find many much more satisfactory standards. Thus it has been suggested by

Sir William Thomson* that the period of vibration of a metallic spring, kept in a hermetically sealed exhausted chamber at a constant temperature, or the period of a particular mode of vibration of a quartz crystal (or other crystal of definite composition) of a specified size and shape and at a given temperature, would be theoretically preferable to the mean solar second, as fulfilling with a much nearer approach to perfection the condition of constancy. Clerk Maxwell † has also suggested as units of time the period of vibration of a gaseous atom of a widely diffused substance easily procurable in a pure form; and the period of revolution of an infinitesimal satellite close to the surface of a globe of matter of the standard density, which may be any density determined by a definite physical condition of any substance, for example the maximum density of water. This period is independent of the size of the globe ‡ and it has been pointed out that this advantage would also be obtained by founding the unit of time on the period of a small simple harmonic vibration of a globe of standard density.

The mass of a gaseous atom, for example that of a sodium or hydrogen atom, and the wave-length of a definite line in the spectrum of an easily obtainable substance, for example one of the D lines in the spectrum of sodium, might be chosen as the units of mass and length. These units would be quite definite since, according to the kinetic theory of gases, the atoms of any one substance are undistinguishable from one another by any physical test.

* *Electricity and Magnetism*, vol. i. p. 3; Thomson and Tait, *Nat. Phil.* vol. i. part i. p. 227 (second edition). † *Ibid.*

‡ For water at the temperature of maximum density, it is approximately 10h. 3m.

We shall denote a numerical time-interval by T. The dimensional formula of time will therefore be $[T]$.

DERIVED UNITS.

Let us suppose that the numerical expression of a physical quantity is given by the equation,

$$N = C \,.\, L_1^{\,l} M_1^{\,m} T_1^{\,n} \,.\, L_2^{\,p} M_2^{\,q} T_2^{\,r} \,.\, \&c.,. \quad . \quad (2)$$

where L_1, L_2, &c., M_1, M_2, &c., T_1, T_2, &c., are numerics for different lengths, masses, and times in terms of a certain chosen unit for each, and C is a numerical multiplier (generally equal to unity) which does not depend on the units adopted. Now let other units of length, mass, and time be chosen and let N be the numerical expression of the same quantity in terms of these units, and L'_1, L'_2, &c., M'_1, M'_2, &c., T'_1, T'_2, &c., those of the lengths, masses, and times. Then we have

$$N' = C \,.\, L'^{\,l}_1 M'^{\,m}_1 T'^{\,n}_1 \,.\, L'^{\,p}_2 M'^{\,q}_2 T'^{\,r} \,.\, \&c. \quad . \quad (3)$$

But by equation (1) $L'^{l} = L^{l} [L]^{l}$, $M'^{m} = M^{m} [M]^{m}$, and so on. Hence (3) becomes,

$$N' = C \,.\, L_1^{\,l} M_1^{\,m} T_1^{\,n} \,.\, L_2^{\,p} M_2^{\,q} T_2^{\,r} \,.\, \&c.\ [L]^{l+p+\&c.} \\ .\, [M]^{m+q+\&c.} \,.\, [T]^{n+r+\&c.} \quad . \quad . \quad . \quad (4)$$

By equation (1) therefore the dimensional formula $[N]$ of the quantity is $[L]^{l+p+\&c.}$. $[M]^{m+q+\&c.}$. $[T]^{n+r+\&c.}$ In accordance with the notation $[N]$, we shall denote this in future by the more convenient expression $[L^{l+p+\&c.}.\ M^{m+q+\&c.}.\ T^{n+r+\&c.}]$.

The numerics $l + p +$ &c., &c., correspond to what

Fourier* called *exposants des dimensions* of the quantities which entered into his analysis, and it is these numerics, not the dimensional formulas, which are properly the "dimensions" of the units. It was pointed out by Fourier that in equations involving the numerics for physical quantities every term must be of the same dimensions in each unit, otherwise some error must have been made in the analysis. This consideration affords in physical mathematics a valuable check on the accuracy of algebraic work.

It is obvious from equation (1) or (4) that the dimensional formula of the product of any number of numerics N_1, N_2 for different physical quantities is the product $[N_1 . N_2$. &c.] of their dimensional formulas, and more generally that the dimensional formula of the product $[N_1^{\mu_1} . N_2^{\mu_2}.]$ &c., of any powers whatever of these expressions, is the product of the same powers of the corresponding dimensional formulas.

We are now prepared to find the dimensional formulas of the various derived units. The process will consist in finding for each quantity the formula corresponding to the right-hand side of (2), and thence deriving according to (4) the proper formula of dimensions. We shall consider first the units of Area, Volume, and Density; then the various dynamical units which are involved in those of electrical and magnetic quantities.

Area. The general formula for the area of any surface can be put in the form CL^2, where L is a numeric expressing a length, and C is a numeric which does not change with the units. Hence by (4) the formula of dimensions for area is $[L^2]$.

Volume. Similarly the formula for a numerical volume

* *Théorie Analytique de la Chaleur*, Chap. II. Sect. IX.

can be written CL^3, and the formula of dimensions is $[L^3]$.

Density. The *Density* of a body is numerically expressed by the numerical mass per unit of volume. We shall denote it by the symbol D.

If the body be of uniform density, the density is obtained by finding the mass contained in any given volume of the body: the ratio of the numerical mass to the numerical volume is the density.

If the body be of varying density, the density at any point is the limit towards which the ratio of the numerical mass contained in an element of volume including the point, to the numerical volume, approaches as the element is taken smaller and smaller. Thus if δV be an element of volume including a point at which the density is D, and δM be the mass of the element, we have

$$D = \text{Limit} \frac{\delta M}{\delta V} = \frac{dM}{dV}.$$

In either case we have for the numerical expression of the volume taken CL^3, and for that of the mass contained in it M. Hence

$$[D] = [ML^{-3}].$$

The *Specific Gravity* of a body is the ratio of the density of the body to the density of the standard substance, and is therefore a numeric independent of the system of units adopted, that is, its dimensional formula is 1. If G denote the specific gravity of a body whose density is D, and D_s be the density of the standard substance,

$$D = G \,.\, D_s.$$

In the French system of units D_s is taken as unity and we have $D = G$. This is one great convenience of

the French units of length and mass; but it is to be remembered that Density and Specific Gravity are essentially different ideas, and only coincide in numerical value when $D_s = 1$.

DYNAMICAL UNITS.

Velocity. The velocity of a body is measured by the numerical length described per unit of time. Its specification involves direction as well as magnitude; but in dealing with the dimensions of velocity we are only concerned with the latter element.

If the velocity is uniform, its numerical expression is the ratio of the numerical distance L traversed to the numerical time-interval T in which it is described.

If the velocity is variable, its numerical expression v at any instant is the value towards which the ratio of the numerical distance δL traversed in an interval of time including the instant, to the numerical time-interval δT, approaches as the interval is taken smaller and smaller. Hence

$$v = \frac{dL}{dT} = \dot{L},$$

where $\dot{L}$ denotes in Newton's fluxional notation the time-rate of variation of L. We shall use this symbol for velocity.

We see that the numerical expression of a velocity is the ratio of a numerical length to the numerical time-interval, and therefore we have

$$[\dot{L}] = [L T^{-1}].$$

As multiplier for a change from mile-minute units to centimetre-second units we have

$$n = 5280 \times 30{\cdot}4797/60 = 2682{\cdot}2136.$$

In statements of amounts of velocities there ought clearly to be a distinct reference to the unit of time: thus the expressions one mile per minute, 88 feet per second, 2682·2136 centimetres per second, are perfectly definite, and express the same velocity, while such a statement as a "velocity of 88 feet" is devoid of meaning.

Acceleration. The acceleration of a body is the rate of change of velocity per unit of time.

Like velocity, acceleration involves in its signification the idea of direction as well as of magnitude; and it is through a want of clear apprehension of this fact that difficulty is found by students in the theory of curvilinear motion.

Let $\delta\dot{L}$ be the velocity given in direction and magnitude which compounded with the velocity $\dot{L}$ which a particle possesses at the beginning of an interval of time δT would give the velocity in direction and magnitude at the end of that interval, then $\delta\dot{L}/\delta T$ is the *average* acceleration during that interval, and the limit towards which this ratio approaches as δT is made smaller and smaller is the true value of the acceleration at the beginning of the interval. That is, we have

$$\text{Acceleration} = \frac{d\dot{L}}{dT} = \ddot{L},$$

where $\ddot{L}$ denotes in the fluxional notation the time-rate of variation of $\dot{L}$, that is, of velocity.

We shall use this symbol for acceleration, and the two dots above the L will serve to recall the double reference to time which is plainly involved in the notion of acceleration. This should be clearly expressed in

statements of amounts of acceleration. Thus such a statement as an acceleration of 981 centimetres per second per second, or 32 feet per second per second, is perfectly definite, while such phrases as an "acceleration of 981 centimetres" or "an acceleration of 32 feet per second," which are often used, are meaningless.

The dimensional formula is

$$[\ddot{L}] = [L\,T^{-2}].$$

For a change from mile-minute units to centimetre-second units,

$$n = 5280 \times 30{\cdot}4797/60^2 = 447{\cdot}0356.$$

Momentum. Taking for simplicity the case of a rigid body moving without rotation, that is, so that each particle of the body has the same velocity at the same instant, the momentum of the body is expressed as the product of the numerics for the mass of the body and its velocity. Hence it is expressed symbolically by $M\dot{L}$. The dimensional formula is therefore

$$[M\dot{L}] = [M\,L\,T^{-1}].$$

Time-Rate of Change of Momentum. If the momentum of the body be not constant, then, since we suppose the mass constant, we must have for the time-rate of variation the expression $M\ddot{L}$, that is, the product of the numerics for the mass and the acceleration. The dimensional formula is therefore

$$[M\ddot{L}] = [M\,L\,T^{-2}].$$

Force (F). A force acting on a body is proportional to the time-rate of change of momentum. Hence the dimensional formula just found is that of force.

According to the system suggested by Gauss, a force is measured by the time-rate of change of momentum, that is, the constant, C, of equation (3) is in this case, as in the other cases we have considered, taken equal to unity. Unit force is therefore that force which acting for unit of time on unit mass produces unit change of velocity, or simply that which produces unit acceleration in unit mass.

When the unit of mass is one pound, the unit of length one foot, and the unit of time one second, then unit force is that force which acting for one second on a pound of matter generates a velocity of one foot per second. This unit force has been called a *poundal.*

The unit force in the C.G.S. system, is that force which, acting for one second on one gramme of matter, generates a velocity of one centimetre per second. To this unit force the name *dyne* has been given.

This method (sometimes called the *kinetic* method) of measuring forces has now superseded, for scientific purposes, the gravitation system formerly in use. In that system the unit of force is the force of gravity on the unit of mass, and has, therefore, different values at different places on the earth's surface, and at different vertical distances from the mean surface level. This substitution of an invariable unit of force, depending only on the standards adopted for length, mass, and time, instead of the former variable unit, is at the foundation of the system of units of measurement established by Gauss. It is to express this fact of invariability with locality and other circumstances that, as already explained, the term "absolute" is used for the unit of force and other derived units in this system.

Work (W). In dynamics *work* is said to be done *by* a force when the place of application of the force receives a component displacement *in the direction in which the force acts*, and the work done by it is equal to the product of the force and the distance through which the place of application of the force has moved in that direction. The time-rate at which work is done by a force at any instant is therefore equal to the product of the force and the component of velocity in the direction of the force at that instant. The work done in overcoming a resistance through a certain distance is equal by this definition to the product of the resistance and the distance through which it is overcome. Among engineers in this country the unit of work generally used is *one foot-pound*, that is, an amount of work equal to that done in lifting a pound vertically against gravity through a distance of one foot. The weight of a pound of matter being generally different at different places, this unit of work is a variable one, and is not used in theoretical dynamics. In the absolute C.G.S. system of units, the unit of work is the work done in overcoming a force of one dyne through a distance of one centimetre, and is called one centimetre-dyne or one *erg*.

In practical electricity 10^7 *ergs* is frequently used as unit of work, and is called a *Joule*.

If F denote a numerical force and L the numerical space-interval through which it has acted, the numeric for work done is FL. Hence we have

$$[W] = [FL] = [ML^2 T^{-2}].$$

Activity (A). The single word *Activity* has been used by Sir William Thomson as equivalent in meaning to "time-rate of doing work," or the rate per unit of time

at which energy is given out by a working system ; and to avoid circumlocutions in what follows we shall frequently use the term in that sense. Among engineers in this country the unit rate of working is *one horse-power*, that is 33,000 foot pounds per minute.

Unit Activity in the C.G.S. system is one *erg* per second. In practical electricity an activity of 10^7 *ergs* per second is frequently employed as unit. This unit has been called a *Watt*.

Since an Activity is measured by the numerical work done per unit of time, its dimensional formula is given by

$$[A] = [M L^2 T^{-3}].$$

Energy (E). When a material system in virtue of stresses between its own parts and those of bodies external to it does work or has work done upon it, in passing from one state to another, it is said to give out or to gain energy. The energy given out or gained is measured by the work so done.

If the change be a change of motion, then, according as energy is given out or gained by the system, it is said to lose or gain *kinetic energy*. If the change be of any other kind, it must be classed under change of configuration, and, according as the system gives out or gains energy, it is usually said to lose or gain *potential energy*.

When we consider the work done by mutual forces between different parts of the same system, a loss of kinetic energy in the system is accompanied by an equal gain of potential energy, and *vice versâ*, so that the total energy of the system remains unchanged in amount. This is the principle called the *Conservation of Energy*.

Energy is measured by the same units as work, and

its dimensional formula is the same as that of work, that is

$$[E] = [ML^2T^{-2}].$$

We shall here, for the sake of illustration, give three examples of the application of dimensional formulas to the solution of problems regarding units. The problems are taken from Professor Everett's *Units and Physical Constants.*

Ex. 1. If the unit of time be the second, the unit density 162 lbs. per cubic foot, and the unit of force the weight of an ounce at a place where the change of velocity g produced by gravity in one second is 32 feet per second, what is the unit of length?

Here the change-ratio by which we must multiply the density of a body in the system of units proposed, to find its density in terms of the pound as unit of mass, and the foot as unit of length, is 162. We have therefore, omitting the brackets in the dimensional formulas,

$$ML^{-3} = 162,$$

where M is the number of pounds equivalent to the unit of mass, and L the number of feet equivalent to the unit of length. Also, it is plain that the unit of force in the proposed system is two foot-pound-second units. Hence we have also, since $T = 1$,

$$MLT^{-2} = ML = 2.$$

By division therefore we get $L^4 = 1/81$ or $L = 1/3$. The unit of length is therefore 4 inches.

Ex. 2. The number of seconds in the unit of time is equal to the number of feet in the unit of length, the unit of force is 750 lb. weight (g being 32), and a cubic foot of

the substance of unit density contains 13,500 ounces. Find the unit of time.

Using M and L as in the last problem, and putting T for the numerical expression of the unit of time in seconds, we have plainly

$$ML^{-3} = \frac{13500}{16}$$

and

$$MLT^{-2} = 750 \times 32.$$

Therefore by dividing, and remembering that $L = T$, we get

$$T^2 = \frac{32 \times 750 \times 16}{13500} = \frac{16^2}{3^2}.$$

That is the unit of time is $5\frac{1}{3}$ seconds.

Ex. 3. When an inch is the unit of length and T seconds the unit of time, the numeric of a certain acceleration is a; when 5 feet and 1 minute are units of length and time respectively, the numeric for the same acceleration is $10a$. Find T.

The change-ratio or value of LT^{-2} for reduction to foot-second units is plainly in the first case $T^{-2}/112$, in the second $5/3600$. We get therefore

$$\frac{1}{12} T^{-2} a = \frac{5}{3600} \times 10a,$$

or

$$T = \sqrt{6}.$$

DERIVED ELECTRICAL UNITS.

I.—ELECTROSTATIC SYSTEM.

Quantity of Electricity [q]. In the electrostatic system of units which is convenient when electrostatic results, independently of their bearing on electromagnetic pheno-

mena, are required, the units of all the other quantities are founded on the following definition of unit quantity of electricity. *Unit quantity of electricity is that quantity which, concentrated at a point at unit distance from an equal and similar quantity, also concentrated at a point, is repelled with unit force when the medium across which the electric action is transmitted is a certain standard insulating medium.* An ideal vacuum is sometimes taken as standard, but we shall suppose at present that the medium is air at temp. 0° C. and at standard atmospheric pressure. We shall call this simply *air*.

This definition is precisely similar to the definition (p. 40 above) of unit magnetic pole which forms the basis of another system of units called the *electromagnetic system*, of much wider and more important application than the electrostatic. Hence by Coulomb's law that (the numerical values of) electric attractions and repulsions are directly as the products of the (numerics for the) attracting and repelling quantities, and inversely as the second power of the (numeric for the) distance between them, if a quantity of positive electricity expressed by q be placed at a point distant L units from an equal quantity of electricity, then the medium being air, the numeric F for the force between them is q^2/L^2.

If the medium across which the electric action is transmitted be some other medium than air, the force between the charges is numerically q^2/KL^2 where K is the numerical measure of a quantity called the *electric inductive capacity*, or usually the *specific inductive capacity* of the medium. This quantity is precisely analogous to the conductivity of a substance for heat * and to magnetic

* See *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. i. chap i. sect. v.

permeability (see p. 350 below). In the ordinary electrostatic system of units this quantity is defined (as at p. 347) so as to have a dimensional formula 1, that is to be a mere numeric.

But we might proceed otherwise and regard K as a quantity of undetermined dimensions as regards the fundamental units, but such that q^2/KL^2 has the dimensions of a force. We may then, in the absence of special reasons for preferring one dimensional formula for K to another, assign its dimensions according to any convenient hypothesis. One such hypothesis is that which forms the basis of the ordinary electrostatic system, namely, that K is, as regards the fundamental units, of zero dimensions, that is has a dimensional formula [1]. But in the ordinary electromagnetic system of units, which has quite a different derivation from the electrostatic, the dimensional formula of K is $[L^{-2}T^2]$ and the numerical value of K depends on the choice made of fundamental units.

We shall in what follows suppose the dimensions of K undetermined, and therefore allow the symbol K expressing it to appear in the dimensional formulas of the other quantities. We shall thus obtain a more general electrostatic system in which the absolute dimensions of the quantities are not settled. From this the ordinary electrostatic system is obtained by simply deleting K.*

The dimensional formula of quantity of electricity is accordingly $[F^{\frac{1}{2}} L K^{\frac{1}{2}}]$ or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} K^{\frac{1}{2}}]$.

Electric Surface Density $[\sigma]$. The density of an electric charge on a surface is measured by the quantity of electricity per unit of area. Therefore $[\sigma]$ is $[q L^{-2}]$ or $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{\frac{1}{2}}]$.

* This method of proceeding is advocated and its advantages pointed out by Prof. A. W Rücker, F.R.S. in a paper on the "Suppressed Dimensions of Physical Quantities," *Phil. Mag.*, Feb. 1889.

Electric Force and Intensity of Electric Field $[f]$. The electric force at any point in an electric field, or, the intensity of the field at that point, is the force with which a unit of positive electricity would be acted on if placed at the point. Hence if the numeric for the quantity of electricity at a point P be q and that of the electric force at that point be f, the numeric F for the force on the electricity is qf, and we have the equations $f = Fq^{-1}$. Therefore $[f]$ is $[Fq^{-1}]$ or $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$.

Electric Potential $[v]$. The difference of electric potential between two points is measured by the work which would be done if a unit of positive electricity were placed at the point of higher potential and made to pass by electric force to the point of lower potential. Hence in transferring q units of electricity through a difference of potential expressed numerically by v, an amount of work is done for which the numeric W is equal to qv. We have therefore $v = Wq^{-1}$, and hence $[v]$ is $[Wq^{-1}]$ or $[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}]$.

Capacity of a Conductor $[c]$. The capacity of an insulated conductor is the quantity of electricity required to charge the conductor to unit potential, all other conductors in the field being supposed at zero potential. Hence denoting the numeric for the capacity of a given conductor by c, those for its charge and potential by q and v, we have $c = qv^{-1}$, and for $[c]$ therefore $[qv^{-1}]$ is $[LK]$. The unit of capacity has therefore the same dimensions as the unit of length provided $[K] = 1$; and the capacity of a conductor is then properly expressed as so many centimetres.

The electrostatic capacity of a conducting sphere is in ordinary electrostatic units numerically equal to the radius of the sphere. A conducting sphere of 1 cm. radius has therefore 1 C.G.S. unit of capacity.

Specific Inductive Capacity $[K]$. The specific inductive capacity of a dielectric has already been virtually defined above, but it is usual to define it as the ratio of the capacity of a condenser, the space between the plates of which is filled with the dielectric, to the capacity of a precisely similar condenser with air as dielectric; or, according to Maxwell's* Theory of Electric Displacement, it is defined as the ratio of the electric displacement produced in the dielectric to the electric displacement produced in air by the same electric force. Thus in the ordinary electrostatic system of units its dimensions are taken as zero, that is, it is simply a numerical coefficient which does not change with the units. Hence in the ordinary electrostatic system $[K] = 1$.

Electric Current $[\gamma]$. An electric current in a conducting wire is measured by the quantity which passes across a given cross-section per unit of time. If q be the numeric for the quantity which has passed in a time for which the numeric is T, then denoting the numeric for the current by γ, we have $\gamma = q/T$, and $[\gamma]$ is $[q\,T^{-1}]$ or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} K^{\frac{1}{2}}]$.

Resistance $[r]$. By Ohm's law the resistance of a conductor is expressed by the ratio of the numeric v for the difference of potential between its extremities to the numeric γ for the current flowing through it. We have therefore $r = v/\gamma$, and $[r]$ is $[v\gamma^{-1}]$ or $[L^{-1}\,T K^{-1}]$.

Conductance (see p. 87 above). The dimensional formula of conductance is plainly $[L\,T^{-1}K]$. Hence in the ordinary electrostatic system its dimensional formula is $[LT^{-1}]$ which is that of velocity. Hence a conductance in electrostatic C.G.S. units is properly expressed in centimetres per second.

* *El. and Mag.* vol. i. 2nd edition, p. 154.

The following illustration of this result has been given by Sir William Thomson. Suppose a spherical conductor charged to a potential v to be connected to the earth by a long thin wire, of which the capacity may be neglected; and let r be the resistance of this wire in electrostatic measure. The current in the wire at the instant of contact is v/r. Now let the sphere diminish in radius at such a constant rate that the potential remains v. The current remains v/r, and the quantity of electricity which flows out in t seconds will be vt/r. If the radius be initially x, and in t seconds has diminished to x', the diminution of capacity is $x - x'$ (p. 346). Hence the loss of charge is $v(x - x')$, and we get $vt/r = v(x - x')$, or $1/r = (x - x')/t$. But $(x - x')/t$ is the velocity with which the radius of the sphere diminishes. The conductivity $1/r$ of the wire is therefore measured numerically by the velocity with which the surface of the sphere must approach the centre, in order that its potential may remain constant when the surface is connected to the earth through the wire.

II.—ELECTROMAGNETIC SYSTEM.

Magnetic Pole or Quantity of Magnetism $[m]$; *Surface density of Magnetism* $[\sigma']$; *Magnetic Force or Magnetic field intensity* $[I]$; *Magnetic potential* $[V]$.

The electromagnetic system of units is based on the unit magnetic pole as defined above (p. 40). This definition is exactly the same as that of unit quantity of electricity on which the electrostatic system is founded; and therefore the purely magnetic quantities here mentioned, which bear the same relations to the unit quantity of magnetism that the corresponding electric quantities

bear to the chosen unit quantity of electricity, have, with the substitution of the magnetic analogue to K, in the electromagnetic system the same dimensional formulas as those just found for the latter quantities in the electrostatic system.

Observations precisely similar to those made above regarding specific inductive capacity apply here regarding its analogue, magnetic inductive capacity, or, as it is frequently called, magnetic permeability. The force between two poles each of strength m at distance L in a medium of magnetic inductive capacity μ is numerically $m^2/\mu L^2$, and hence $[m] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \mu^{\frac{1}{2}}]$. In the ordinary electromagnetic system μ is defined (see p. 350 below) so as to be a mere numeric. We shall not here make this assumption, but allow μ to appear in the formulas, and its dimensions may be afterwards assigned.

By simple deletion of μ from the dimensional formulas they become those for the ordinary electromagnetic system in which $[\mu] = 1$.

Magnetic Moment $[\mathfrak{m}]$. The numeric $\mathfrak{m}$, the magnetic moment of a uniformly magnetized bar-magnet, is the product of the numerics for the strength of either pole and the length of the magnet. Hence we have $[\mathfrak{m}] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}] . [L] = [M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1} \mu^{\frac{1}{2}}]$.

Intensity of Magnetization $[\nu]$. The intensity of magnetization of any portion of a magnet is measured by the magnetic moment of that portion per unit of volume. Hence, if ν denote the numeric for the intensity of magnetization of a uniformly magnetized magnet, the numerics for the magnetic moment and volume of which are $\mathfrak{m}$ and $A L^3$, we have

$$\nu = \frac{\mathfrak{m}}{A L^3}, \text{ and } [\nu] = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}].$$

It is plain that the intensity of magnetization of a uniformly and longitudinally magnetized bar is equal to the surface density of the magnetic distribution over the ends of the bar, and therefore intensity of magnetization has the same dimensional formula as magnetic surface density.

*Magnetic Permeability** [μ]. The magnetic permeability of an inductively magnetized substance, or its magnetic inductive capacity, is, as has already been stated, the analogue in magnetism of specific inductive capacity of a dielectric in electricity, and of the conductivity of a body for heat in heat conduction. It is usual to define it as measured at any point by the ratio of the force which a unit pole would there experience if placed in a narrow crevasse, cut in the substance so that its walls are at right angles to the direction of magnetization, to the force which it would experience if placed in a narrow crevasse, the walls of which are parallel to the direction of magnetization. In the first case we must suppose, in consequence of the formation of the crevasse, a distribution of blue magnetism over one of its walls, and a similar distribution of red magnetism over the opposite wall. These magnetic distributions are exactly equal and opposite to the distributions on the sides of the portion of the substance removed to form the crevasse. If we suppose the substance uniformly magnetized, the density of this distribution will be uniform on both walls, and if we denote the density in one by σ', that on the other will be denoted by $-\sigma'$. A unit blue magnetic pole placed in the crevasse would be repelled from the blue side and attracted towards the red with a force in each case equal

* See Reprint of Papers on *Electrostatics and Magnetism*, by Sir W. Thomson, p. 484.

to $2\pi\sigma'$;* and therefore would be acted on towards the red side by a total force of $4\pi\sigma'$. Besides this force due to the magnetic distribution in the crevasse, the pole is acted on by the resultant of the forces due to the other magnetic distributions of the system. If we suppose the substance isotropic as to magnetic quality, that is, to have the same magnetic quality in different directions, this latter force will be in the same direction as the former. Calling it f, we have for the total force acting on the pole towards the red side of the crevasse the expression $f + 4\pi\sigma'$.

If, on the other hand, the pole were placed in the same position but in a narrow crevasse cut with its walls parallel to the direction of magnetization of the substance, the force acting on it would be simply f. Hence, if we call μ the magnetic permeability of the substance, we have $\mu = 1 + 4\pi\sigma'/f$.

It is clear that this mode of defining μ makes it in the ordinary electromagnetic system a mere numeric, that is its dimensional formula is $[\mu]$ 1.

Magnetic Susceptibility. The ratio σ'/f is called the

* Imagine a material particle of unit mass (such a particle being here defined as a particle which placed at unit distance from an equal particle would be attracted with unit force) situated at a point on the axis of a thin uniform material disk of radius r. Let the mass of the disk per unit of area be σ, and the distance of the particle from the disk measured along the axis a. The attraction on the particle of a small element of the disk of area dS at a distance x from the centre is $\sigma ds/(a^2 + x^2)$ and the component resolved along the axis is $a\sigma dS/(a^2 + x^2)^{\frac{3}{2}}$. Hence the attraction in this direction due to a narrow ring of radius x and breadth dx is $2\pi\sigma a x dx/(a^2 + x^2)^{\frac{3}{2}}$, and that of the whole disk therefore

$$2\pi\sigma a\int_0^r \frac{x\,dx}{(a^2 + x^2)^{\frac{3}{2}}} = 2\pi\sigma\left(1 - \frac{a}{\sqrt{a^2 + r^2}}\right).$$

If the distance a of the particle from the disk be very small in comparison with the radius r of the disk, this expression for the total attraction becomes simply $2\pi\sigma$. We have only to substitute a unit magnetic pole for the unit particle and a distribution of imaginary magnetic matter of surface density σ' for the material disk, and we have the result used in the text.

magnetic susceptibility of the substance. Its dimensional formula is therefore also 1 in the ordinary electromagnetic system, that is magnetic susceptibility is in that system a mere numeric.

Current Strength [Γ]. By the theory of electromagnetic action stated above in p. 44, and the definition of unit current (3), p. 47, we have, for any actual case of a magnetic pole placed at the centre of a circle of wire carrying a current, the equation $\Gamma = FL/2\pi m$, where F, L, m, and Γ are the numerics respectively for the force acting on the pole, the radius of the circle, the strength of the pole, and the strength of the current. Hence $[\Gamma] = [FLm^{-1}] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]$.

Quantity of Electricity [Q]. The numeric Q for the quantity of electricity conveyed in T seconds by a current the numeric for the strength of which is C is equal to CT. Hence $[Q] = [CT] = [M^{\frac{1}{2}} L^{\frac{1}{2}} \mu^{-\frac{1}{2}}]$.*

Electric Potential, or Electromotive Force [V]. As above (p. 346), but using in this case the symbol V for a numerical difference of potential, we get $W = VQ$. Thus we have $[V] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \mu^{\frac{1}{2}}]$.

Electrostatic Capacity [C]. Using for a numerical capacity in electromagnetic units the symbol C, we find, by the same process as in p. 346, the equation $C = Q/V$, $[C] = [L^{-1} T^2 \mu^{-1}]$.

Resistance [R]. Using here R to denote a numerical

* We might pass in the electrostatic system from the dimensional formula of unit current to that of unit quantity of magnetism, precisely as we pass here in the electromagnetic system from the dimensional formula of unit quantity of magnetism to that of unit current, and we should find for the dimensional formula sought that here obtained $[M^{\frac{1}{2}} L^{\frac{1}{2}} K^{-\frac{1}{2}}]$ as might be inferred at once. From this the formulas in the electrostatic system for all the other magnetic quantities might be found.

resistance, we get as formerly $R = V/C$, and therefore $[R] = [L T^{-1} \mu]$.

Thus if $[\mu] = 1$, the dimensional formula for resistance is the same as that for velocity, and therefore a resistance in ordinary electromagnetic units is properly expressed as a velocity, and accordingly, in C.G.S. units, as so many centimetres per second. This fact is directly shown by the following illustration, due to Sir William Thomson. Let the rails of the ideal machine, described in p. 67, be supposed to run horizontally at right angles to the magnetic meridian, and let their plane be vertical. Let a tangent galvanometer be included in the wire connecting the rails. The slider when moved along the rails will cut the lines of the earth's horizontal force, the intensity of which in electromagnetic measure we have denoted by H. If the slider have a length L and be moved with a velocity v, the electromotive force developed will be HLv. If R be the total resistance in circuit, C the current flowing, r the mean radius of the galvanometer coil, and L' the length of wire in the coil, we have by (2), p. 28, $C = Hr^2/L' \tan\theta$. But by Ohm's law $C = HLv/R$. Hence $HLv/R = Hr^2/L' \tan\theta$, or

$$R = \frac{LL'v}{r^2 \tan\theta}.$$

Now we may suppose the radius r of the coil so taken that $r^2 = LL'$, and that the slider is moved at such a speed, v, that the deflection of the needle is 45°. Under these conditions we get $R = v$. The resistance R of the circuit is therefore measured in electromagnetic units by the velocity with which the slider must be moved, so that the deflection of the needle of the tangent galvanometer may be 45°.

Coefficient of Self-Induction (*or Inductance*). Denoting by l (instead of L to avoid confusion) the inductance of a circuit, the current in which is Γ, we have (p. 282) $ld\Gamma/dt$ for the electromotive force of self-induction. Hence $ld\Gamma/dt$ has the same dimensional formula as electromotive force, that is $[l\Gamma T^{-1}] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \mu^{-\frac{1}{2}}]$, and therefore $[l] = [L\mu]$

Coefficient of Mutual Induction. If k be the co-efficient of mutual induction between two circuits, Γ the current in one of them, then the electromotive force in the other circuit due to mutual induction is $kd\Gamma/dt$. Hence by the same process as before we get $[k] = [L\mu]$.

A co-efficient of self- or of mutual induction is therefore in ordinary electromagnetic measure in dimensions simply a length, and in C.G.S. units is properly expressed as so many centimetres.

But if resistance is taken in terms of the true ohm which is 10^9 cms. (or nearly one earth-quadrant) per second, the corresponding unit of induction is 10^9 cms. If the legal or any other ohm is used, the unit of induction is that length which replaces 10^9 cms. in the definition of the ohm.

For the unit of induction defined by *any* ohm Profs. Ayrton and Perry have proposed the name *secohm*. The Paris Congress has however adopted the name *quadrant*. Plainly this can only in strictness be applied in connection with the true ohm.

We have now investigated the dimensional formulas of the absolute units of all the principal electric and magnetic quantities in the electrostatic system, or in the electro-magnetic system, according as each quantity is generally measured in practice. Each may, however, be expressed either in electrostatic or in electromagnetic

units, and we give the following table of dimensional formulas for all the quantities in both systems.

In Tables II. and III. K and μ have been introduced into the formulas as stated above, pp. 345, 349. The ordinary electrostatic and electromagnetic systems are obtained by supposing K and μ each unity.

One advantage of thus exhibiting the dimensions is that it enables electrostatic and electromagnetic quantities to be regarded as of the same absolute dimensions, since K and μ, not being fixed as to dimensions, can, unless restricted by definition, have dimensions assigned to them which fulfil this condition. For example, as suggested by Professor G. F. Fitzgerald,* each may be taken as having the dimensions $[T L^{-1}]$. Another advantage is that problems in which passage from one set of units to the other is involved are solved with greater ease from first principles (See Professor Rücker's paper *loc cit.*)

FUNDAMENTAL UNITS.

Quantity.	Dimensional Formula.
Length	$[L]$
Mass	$[M]$
Time	$[T]$

DERIVED UNITS.

I. Dynamical Units.

Velocity	$[L\ T^{-1}]$
Acceleration	$[L\ T^{-2}]$
Force	$[M\ L\ T^{-2}]$
Work / Energy	$[M\ L^2\ T^{-2}]$

* *Phil. Mag.*, April 1889.

NOTE.—Col. A below with K deleted gives the ordinary electrostatic formulas, Col. B with μ deleted gives the ordinary electromagnetic formulas.

II. Electric Units.

	A. In terms of L, M, T, K.	B. In terms of L, M, T, μ.
Quantity of Electricity	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}]$
Surface density of Electricity Electric Displacement	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\mu^{-\frac{1}{2}}]$
Electric Force, or Intensity of Electric Field	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}]$
Electric Potential Electromotive Force	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}]$
Specific Inductive Capacity	$[K]$	$[L^{-2}T^{2}\mu^{-1}]$
Electrostatic Capacity	$[LK]$	$[L^{-1}T\mu^{-1}]$
Current Strength	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Resistance	$[L^{-1}TK^{-1}]$	$[LT^{-1}\mu]$

III. Magnetic Units.

Quantity of Magnetism, or Magnetic Pole	$[M^{\frac{1}{2}}L^{\frac{1}{2}}K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Surface density of Magnetism	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Magnetic Moment	$[M^{\frac{1}{2}}L^{\frac{3}{2}}K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Intensity of Magnetization	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}K^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Magnetic Force or Intensity of Magnetic Field	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Magnetic Potential	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Magnetic Inductive Capacity Magnetic Susceptibility	$[L^{-2}T^{2}K^{-1}]$	$[\mu]$
Coefficient of Self-Induction Coefficient of Mutual Induction	$[L^{-1}T^{2}K^{-1}]$	$[L\mu]$

As an example of the use of dimensional formulas we may find the multiplier for the reduction of numerics for magnetic field intensities given in terms of British foot-grain-second units to the corresponding numerics in terms of C.G.S. electromagnetic units. Let H be the numerical intensity in terms of British units, H' the numerical intensity in C.G.S. units. We have, by equation (4), p. 333,

$$H' = H[H] = H[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}].$$

Since 1 gramme = 15·43235 grains, and 1 centimetre = 1/30·47945 foot, we have

$$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}] = \left(\frac{1}{15·43235 \times 30·47945}\right)^{\frac{1}{2}} = \frac{1}{21·688}.$$

The earth's horizontal force is given as 3·92 in British units at Greenwich for 1883. We get therefore

$$H' = 3·92 \frac{1}{21·688} = ·18075, \text{ in C.G.S. units.}$$

Units adopted in Practice.

In practical work the resistances and electromotive forces occurring to be measured are usually so great that if the absolute electromagnetic C.G.S. units were used, the resulting numerics would be inconveniently large; while, on the other hand, capacities are generally so small that their numerics in C.G.S. units would be only very small fractions. Accordingly certain multiples of the C.G.S. units of resistance and electromotive force, and a submultiple of that capacity have been chosen for use in practice. The first two, the ohm and the volt, together with the practical units of current and quantity, the ampere and the coulomb, have been explained above

(Chap. V.). The practical unit of electrostatic capacity is called the *farad*, and may be defined as the capacity of a condenser which, when charged by an electromotive force of one volt, has a charge of one coulomb. If C be the numerical capacity of such a condenser in C.G.S. electromagnetic units of capacity, we have $C = 10^{-1}/10^{8} = 10^{-9}$; or one farad is equivalent to 10^{-9} C.G.S.

In some cases, when the quantities to be expressed are very large, units one million times greater than the chosen practical units are employed. These are denoted by the names of the corresponding practical units with *mega* (great) prefixed. On the other hand, for the expression of very small quantities, units one million times smaller than the practical units are sometimes used, and are denoted by the corresponding names of the practical units with *micro* (small) prefixed.

Such units are however rarely employed, with the exception of the *megohm*, used for expressions of resistances of insulating substances, and the *microfarad*, which is really the most convenient unit for expressions of capacities. A megohm is a velocity of 10^{15} centimetres per second; one C.G.S. unit of capacity is equivalent to 10^{15} microfarads.

The practical units which have been adopted may be considered as belonging to an absolute system, based on a unit of length equivalent to one thousand million (10^{9}) centimetres (approximately the length of one quadrant of the earth's polar circumference), a unit of mass equivalent to one one-hundred-millionth of a milligramme; or 10^{-11} gramme, and the second as unit of time. The verification of this in the different cases will furnish examples of the use of dimensional formulas.

For example, let us find what the expressions of

resistances and electromotive forces in C.G.S. units become when these new units of length and mass are substituted for the centimetre and the gramme. Let R be the numeric of a resistance in C.G.S. units, and R' its numeric in terms of the new units. We have, by (4) p. 333 above,

$$R' = R\,[L\,T^{-1}] = R \times \frac{1}{10^9}$$

since the unit of time remains unchanged. One ohm is therefore equivalent to 10^9 C.G.S. units of resistance, that is 10^9 centimetres per second.

Again let V be the expression of an electromotive force in C.G.S. electromagnetic units, E' its expression in terms of the new units. The dimensional formula for electromotive force is $[M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}]$. We have therefore

$$E' = E\,[M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}].$$

We have only to consider what $[M^{\frac{1}{2}}\,L^{\frac{3}{2}}]$ becomes. This is plainly $(1/10^{-11})^{\frac{1}{2}} \times (1/10^9)^{\frac{3}{2}}$ or $1/10^8$. Hence

$$E' = E \times \frac{1}{10^8},$$

that is, one volt is equivalent to 10^8 C.G.S. units of electromotive force.

The following table gives the numerics for the various practical units in terms of C.G.S. units :—

Name of Quantity.	Practical Unit.	Equivalent in C.G.S. Units.
Resistance	Ohm	10^9
Electromotive Force	Volt	10^8
Current Strength	Ampere	10^{-1}
Quantity of Electricity	Coulomb	10^{-1}
Electrostatic Capacity	Farad	10^{-9}
	Microfarad	10^{-15}

We have seen above that if Q, Q' be the numerics of two quantities the dimensional formula of Q'/Q is $[Q']/[Q]$, and this of course applies to the expressions of the same quantity in two different systems of units. Thus if q denote the numerical expression of a quantity of electricity in electrostatic units, and Q that of the same quantity in electromagnetic units, the same fundamental units being employed in both cases, the dimensional formula of q/Q is $[q]/[Q]$. But from the table (p. 357) we have $[q]=[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}]$ and $[Q]=[M^{\frac{1}{2}} L^{\frac{1}{2}}]$. The dimensional formula of q/Q is thus the same as that of velocity, that is to say q/Q is equal to a *certain definite* velocity, the numerical expression of which depends on the fundamental units of length and time employed. In other words the number of *electrostatic* units of electricity which is equivalent to one *electromagnetic* unit is numerically equal to this velocity.

The same velocity is derivable from the ratios of the numerical values of any of the other electrical or magnetic quantities in the two systems of units. For instance, if e be the numerical value of an electromotive force in electrostatic units, and E that of the same electromotive force in electromagnetic units, we have for the dimensional formula of e/E the value $[e]/[E]=[M^{\frac{1}{2}} L^{\frac{1}{2}}]/[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}]=[L^{-1} T]$. The ratio e/E has thus the dimensional formula of the reciprocal of a velocity, and therefore E/e, or which is the same, the number of electromagnetic units equivalent to one electrostatic unit of electromotive force, is properly expressed as a certain definite velocity. It is easy to see that this velocity is identical with the former. For if q and Q be the numerical values in the two systems of a certain quantity of electricity, then since e and E denote the same electromotive force, the work eq must be

numerically equal to the work $E\,Q$. We get therefore $E/e = q/Q$, that is, the two velocities are the same.

By taking the more general dimensional formulas given in the table (p. 357) we find that

$$\frac{[q]}{[Q]} = [K^{-\frac{1}{2}}\mu^{-\frac{1}{2}}]$$

when $[q]$, $[Q]$ refers to the ordinary systems. Hence the product $K^{-\frac{1}{2}}\mu^{-\frac{1}{2}}$ has the dimensions of a velocity. It *is* in fact the velocity q/Q above referred to*

Denoting this velocity by v,† we get for the various quantities the following relations. The numerator of the ratio on the left of each equation denotes the numeric of the quantity in electrostatic units, the denominator the numeric of the same quantity in electromagnetic units.

A given	Quantity of Electricity	$\frac{q}{Q} = v$
,,	Current	$\frac{\gamma}{\Gamma} = v$
,,	Electromotive Force	$\frac{e}{E} = \frac{1}{v}$
,,	Electrostatic Capacity	$\frac{c}{C} = v^2$
,,	Resistance	$\frac{r}{R} = \frac{1}{v^2}$

Therefore if q and Q, e and E, or the numerics for any other given quantity, be determined in the two systems of units, the value of v can be at once obtained. Experiments of this kind have been made by Maxwell, Sir W. Thomson,

* See Maxwell, *El. and Mag.* chap. xx. or the author's larger treatise, vol. ii.

† For illustrations of the physical meaning of v see Maxwell's *Electricity and Magnetism*, vol. ii. chap. xix.

Weber, Ayrton and Perry, J. J. Thomson, H. A. Rowland, E. B. Rosa, and others,* with the result that $v = 3 \times 10^{10}$ centimetres per second approximately, or very nearly the velocity of light in air as deduced from experiments made by the methods of Foucault and Fizeau. According to Maxwell's Electromagnetic Theory of Light (*Electricity and Magnetism*, vol. ii., chap. xx.) this relation should hold, and thus the theory is so far confirmed.

Full information regarding experiments for the determination of v will be found in Maxwell's *Electricity and Magnetism*, vol. ii., chap. xix., and an account of the principal determinations made down to the present time will be found in the author's larger treatise,† (vol. ii.), which contains a much more detailed and complete account of electric and magnetic measurements than it is the object of this book to give. But to Maxwell's work also we refer the reader who wishes to obtain a complete account of the mathematical theory of electricity and magnetism measurements. He may consult also portions of Sir William Thomson's *Reprint of Papers on Electrostatics and Magnetism*; Mascart and Joubert's *Leçons sur l'Electricité et le Magnétisme*;‡ and Prof. Chrystal's articles on Electricity and Magnetism,§ which contain an admirable digest of the whole mathematical theory together with much valuable information as to experimental results.

* An account of the principal determinations of this quantity with the results will be found in the author's *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vol. ii.

† The *Theory and Practice of Absolute Measurements in Electricity and Magnetism*, vols. i. and ii. Macmillan & Co.

‡ An English translation of this work by Prof. Atkinson has lately been published.

§ *Encyclopædia Britannica*, New Edition.

APPENDIX.

APPENDIX.

NOTE.

Recommendations of the Paris Congress and the British Association as to Practical Electrical Units.

At the meeting of the Electrical Congress held in Paris in 1884, it was decided to adopt for the present, as practical unit of resistance, a resistance equal to that of a uniform column of mercury one square millimetre in section, 106 centimetres in length, and throughout at the temperature 0° C. The mercury column thus specified expressed approximately and in round numbers the value of the ohm according to the latest and most accurate experiments. It was resolved to give this unit the name *Legal Ohm.*

The Congress also arrived at certain conclusions regarding the practical units of current, electromotive force, quantity of electricity, and electrostatic capacity, as follows:—

(1) That the unit of current should be called the *Ampere*, and be defined as $\frac{1}{10}$ of a C.G.S. electromagnetic unit of current.

(2) That the *Volt* or practical unit of electromotive force, or difference of potentials, should be defined as the electromotive force required to maintain a current of one ampere through a resistance of one ohm.

(3) That the unit quantity of electricity should be called the *Coulomb*, and be defined as equal to the quantity of electricity transferred by a current of one ampere in one second.

(4) That the *Farad* or practical unit of capacity should be the capacity of a conductor which is charged to a potential of one volt by one coulomb of electricity.

The British Association at its meeting in 1886 agreed that the Committee on Electrical Standards should recommend to Her Majesty's Government :—

(1) "To adopt for a term of ten years the Legal Ohm of the Paris Congress as a legalized standard sufficiently near to the absolute Ohm for commercial purposes.

(2) "That at the end of the ten years' period the Legal Ohm should be defined to a closer approximation to the absolute Ohm.

(3) "That the resolutions of the Paris Congress with respect to the Ampere, the Volt, the Coulomb, and the Farad, be adopted.

(4) "That the Resistance Standards belonging to the Committee of the British Association on Electrical Standards, now deposited at the Cavendish Laboratory at Cambridge, be accepted as the English Legal Standards conformable to the accepted definition of the Paris Congress."

At the meeting of the Electrical Congress held at Paris in August, 1889, resolutions were carried of which the following is an English equivalent adopted by the British Association Committee on Electrical Standards (Report, Newcastle Meeting, September, 1889):—

(1) The name of the practical unit of work shall be the *joule*.

The joule is equivalent to 10^7 C.G.S. units of work. It is the energy expended during 1 second by a current of 1 ampere when traversing a resistance of 1 ohm. [See above, p. 77.]

(2) The name of the practical unit of power shall be the *watt*.

The watt is the rate of working of a machine performing 1 joule per second. The power of a machine could naturally be expressed in kilowatts instead of in horse-power. [See above, p. 78.]

(3) The name of the practical unit of light intensity shall be the *candle*.

The candle is equal to the twentieth part of the absolute standard of light as defined by the International Conference of 1884.*

(4) The name of the practical unit of induction shall be the "quadrant." 1 quadrant is equal to 10^9 cms.

(5) The *period* of an alternating current is the duration of a complete oscillation.

(6) The *frequency* of an alternating current is the number of complete oscillations per second.

(7) The *mean current* through a circuit is the time-average of the current. [It is defined by

$$\text{mean current} = \frac{1}{T/2}\int_0^{T/2} C\,dt,$$

C being the current at each instant of the time $T/2$ (T = period of the current) which is reckoned from an instant at which the current is increasing through zero.

* This standard is what was defined by that Conference as the *practical unit of white light*, viz. the total quantity of light emitted by a square centimetre of melted platinum at the temperature of solidification.

The mean current is thus C_m of p. 283 above, there also called the "mean current."]

(8) The *effective current* is the square root of the time-average of the square of the current. Thus

$$\text{effective current} = \left\{ \frac{1}{T}\int_0^T C^2\,dt \right\}^{\frac{1}{2}}.$$

[The effective current is therefore C' of p. 284 above.]

(9) The *effective electromotive force* is the square root of the time-average of the square of the electromotive force. Thus

$$\text{effective electromotive force} = \left\{ \frac{1}{T}\int_0^T E^2\,dt \right\}^{\frac{1}{2}}.$$

E being the actual electromotive force at each instant of the time T.

(10) The *impedance* is the factor by which the effective current must be multiplied to give the effective electromotive force. Thus, in the case of a circuit of R ohms and self-induction L quadrants in which a simple harmonic electromotive force of frequency $n/2\pi$ is acting,

$$\text{impedance} = \sqrt{R^2 + n^2 L^2}.$$

(11) In an accumulator the positive pole is that which is connected with the positive pole of the machine when charging, and from which the current passes into the external circuit when discharging.

RESOLUTIONS OF THE BOARD OF TRADE COMMITTEE ON ELECTRICAL STANDARDS.

(From a Report to the President of the Board of Trade, dated November 29, 1892. These resolutions the Committee desire to substitute for those contained in a previous report of date July, 1891, with the view of obtaining international agreement as to Electrical Standards):—

RESOLUTIONS.

(1) "That it is desirable that new denominations of standards for the measurement of electricity should be made and approved by Her Majesty in Council as Board of Trade standards.

(2) "That the magnitudes of these standards should be determined on the electromagnetic system of measurement with reference to the centimetre as unit of length, the gramme as unit of mass, and the second as unit of time, and that by the terms centimetre and gramme are meant the standards of those denominations deposited with the Board of Trade.

(3) "That the standard of electrical resistance should be denominated the ohm, and should have the value 1,000,000,000 in terms of the centimetre and second.

(4) "That the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross sectional area, and of a length of 106·3 centimetres, may be adopted as one ohm.

(5) "That a material standard, constructed in solid metal, should be adopted as the standard ohm, and should from time to time be verified by comparison with a column of mercury of known dimensions.

(6) "That for the purpose of replacing the standard, if lost, destroyed, or damaged, or for ordinary use, a limited number of copies should be constructed which should be periodically compared with the standard ohm.

(7) "That resistances constructed in solid metal should be adopted as Board of Trade standards for multiples and submultiples of the ohm.

(8) "That the value of the standard of resistance constructed by a Committee of the British Association for the Advancement of Science in the years 1863 and 1864, and known as the British Association unit, may be taken as ·9866 of the ohm.

(9) "That the standard of electrical current should be denominated the ampere, and should have the value one-tenth (0·1) in terms of the centimetre, gramme, and second.

(10) "That an unvarying current which, when passed through a solution of nitrate of silver in water, in accordance with the specification attached to this report, deposits silver at the rate of 0·001118 of a gramme per second, may be taken as the current of one ampere.*

(11) "That an alternating current of one ampere shall mean a current such that the square root of the time-average of the square of its strength at each instant in amperes is unity.

(12) "That instruments constructed on the principle of the balance, in which, by the proper disposition of the conductors, forces of attraction and repulsion are produced, which depend upon the amount of current passing, and are balanced by known weights, should be adopted as the Board of Trade standards for the measurement of current, whether unvarying or alternating.

(13) "That the standard of electrical pressure should be denominated the volt, being the pressure which, if steadily applied to a conductor whose resistance is one ohm, will produce a current of one ampere.

(14) "That the electrical pressure at a temperature of 15° centigrade between the poles or electrodes of the voltaic cell known as Clark's cell, prepared in accordance with the specification attached to this report, may be taken as not differing from a pressure of 1·434 volts, by more than one part in 1000.*

(15) 'That an alternating pressure of one volt shall mean a pressure such that the square root of the time-average of the square of its value at each instant in volts is unity.

(16) "That instruments constructed on the principle of Lord Kelvin's quadrant electrometer used idiostatically, and, for high-pressures, instruments on the principle of the balance, electrostatic forces being balanced against a known weight, should be adopted as Board of Trade standards for the measurement of pressure, whether unvarying or alternating."

* For full particulars as to carrying out the electrolysis of silver, and the construction and use of Clark's cell, see Chapter VII., p. 151.

TABLE I.

Cross-section of Round Wires, with Resistance, Conductivity, and Weight of Hard-Drawn Pure Copper Wires, according to the New Standard Wire Gauge Legalized by Order in Council, August 23, 1883.

Temperature 15° Cent.

No.	Diameter.		Area of Cross-section.		Resistance.		Conductivity.		Weight. (Density=8·95)	
	Ins.	Cms.	Sq. ins.	Sq cms.	Legal Ohms per Yard.	Legal Ohms per Metre.	Yards per Legal Ohm.	Metres per Legal Ohm.	Lbs. per Yard.	Grms. per Metre
0000000	·500	1·270	·1963	1·267	·000125	·000136	8055	7365	2·285	1134
000000	·464	1·179	·1690	1·091	·000144	·000157	6937	6343	1·970	976·3
00000	·432	1·097	·1466	·946	·000166	·000182	6013	5498	1·706	846·3
0000	·400	1·016	·1257	·811	·000194	·000213	5054	4714	1·463	725·6
000	·372	·945	·1087	·701	·000225	·000245	4459	4077	1·265	627·6
00	·348	·884	·0951	·614	·000256	·000280	3901	3568	1·107	549·6
0	·324	·823	·0824	·532	·000296	·000323	3384	3093	·960	476·1
1	·300	·762	·0707	·456	·000345	·000377	2899	2652	·823	408·1
2	·276	·701	·0598	·386	·000408	·000446	2454	2244	·696	345·4
3	·252	·640	·0499	·322	·000489	·000536	2046	1871	·581	288·0
4	·232	·589	·0423	·273	·000577	·000631	1734	1586	·492	244·1
5	·212	·538	·0353	·228	·000691	·000756	1451	1324	·411	203·8
6	·192	·488	·0290	·187	·000842	·000921	1197	1086	·337	166·8
7	·176	·447	·0243	·157	·00100	·00110	988	912	·283	140·5
8	·160	·406	·0201	·130	·00122	·00135	824	748	·234	116·1
9	·144	·366	·0163	·105	·00149	·00164	669	611	·190	94·0
10	·128	·325	·0129	·0830	·00190	·00208	528	482	·150	74·3
11	·116	·295	·0106	·0682	·00230	·00252	434	396	·123	61·0
12	·104	·264	·00849	·0548	·00287	·00314	348	318	·0989	49·0
13	·092	·234	·00665	·0429	·00367	·00402	273	250	·0774	38·4
14	·080	·203	·00503	·0324	·00485	·00530	206	188	·0585	29·0
15	·072	·183	·00407	·0263	·00599	·00657	167	153	·0474	23·5
16	·064	·163	·00322	·0208	·00752	·00839	132	120	·0374	18·6
17	·056	·142	·00246	·0159	·0099	·0108	101	91·5	·0287	14·2
18	·048	·122	·00181	·0117	·0135	·0147	74·2	67·[illegible]	·0211	10·4
19	·040	·102	·00126	·00811	·0194	·0212	51·6	47·1	·0146	7·26
20	·036	·0914	·00102	·00657	·0239	·0262	41·8	38·2	·0118	5·88
21	·032	·0813	·000804	·00519	·0304	·0331	32·9	30·1	·00936	4·64
22	·028	·0711	·000616	·00397	·0396	·0433	25·3	23·[illegible]	·00717	3·56
23	·024	·0610	·000452	·00292	·0539	·0589	18·5	17·0	·00526	2·61
24	·022	·0559	·000380	·00245	·0642	·0701	15·6	14·3	·00443	2·19
25	·020	·0508	·000314	·00203	·0778	·0849	12·8	11·8	·00366	1·80

Cross-section of Round Wires, with Resistance, Conductivity, and Weight of Hard-Drawn Pure Copper Wires, according to the New Standard Wire Gauge Legalized by Order in Council, August 23, 1883.

Temperature 15° Cent.

No.	Diameter.		Area of Cross-section.		Resistance.		Conductivity.		Weight. (Density=8·95)	
	Ins.	Cms.	Sq. ins.	Sq. cms.	Legal Ohms per Yard.	Legal Ohms per Metre.	Yards per Legal Ohm.	Metres per Legal Ohm.	Lbs. per Yard.	Grms. per Metre.
26	·018	·0457	·000254	·00164	·0958	·105	10·4	9·54	·00296	1·47
27	·0164	·0417	·000211	·00136	·116	·123	8·65	7·93	·00246	1·22
28	·0148	·0376	·000172	·00111	·141	·155	7·07	6·45	·00200	·893
29	·0136	·0345	·000145	·000937	·168	·183	5·95	5·45	·00169	·839
30	·0124	·0315	·000121	·000779	·202	·221	4·86	4·53	·00141	·697
31	·0116	·0295	·000106	·000682	·230	·252	4·34	3·96	·00123	·610
32	·0108	·0274	·0000916	·000591	·266	·291	3·75	3·44	·00107	·529
33	·0100	·0254	·0000785	·000507	·311	·339	3·22	2·94	·000914	·453
34	·0092	·0234	·0000665	·000429	·367	·402	2·73	2·50	·000774	·384
35	·0084	·0213	·0000554	·000358	·440	·481	2·27	2·08	·000645	·320
36	·0076	·0193	·0000454	·000293	·540	·587	1·86	1·70	·000548	·262
37	·0068	·0173	·0000363	·000234	·672	·736	1·49	1·36	000423	·210
38	·0060	·0152	·0000283	·000182	·862	·944	1·16	1·04	·000329	·163
39	·0052	·0132	·0000212	·000137	1·15	1·26	·870	·796	·000247	·123
40	·0048	·0122	·0000181	·000117	1·32	1·47	·759	·679	·000211	·104
41	·0044	·0112	·0000152	·0000981	1·60	1·75	·624	·570	·000177	·0373
42	·0040	·0102	·0000126	·0000811	1·94	2·13	·516	·471	·000146	·0726
43	·0036	·00914	·0000102	·0000657	2·39	2·62	·418	·382	·000118	·0588
44	·0032	·00813	·00000804	·0000519	3·04	3·32	·330	·301	0000936	·0464
45	·0028	·00711	·00000616	·0000397	3·96	4·33	·253	·230	·0000717	·0356
46	·0024	·00610	·00000452	·0000292	5·39	5·90	·185	·170	·0000527	·0261
47	·0020	·00508	00000314	·0000203	7·76	8·49	·128	·118	0000366	·0181
48	·0016	·00406	·00000201	·0000130	12·20	13·3	·0824	·0754	·0000234	·0116
49	·0012	·00305	·00000113	·00000730	21·6	23·5	·0464	·0425	·0000132	·00653
50	·0010	·00254	·000000785	·00000507	31·1	33·9	·0322	·029	·00000914	·00453

Note.—The resistances and conductivities in Tables I. and II. are calculated by taking $1·624 \times 10^{-6}$ Legal Ohm as the resistance at 0° C., between the ends of a hard-drawn copper wire 1 cm. long and 1 sq. cm. in cross-section This agrees with Matthiessen and Hockin's result (*B. A. Rep.*, 1864, and *Phil. Mag.*, vol. xxix., 1865) of ·1469 B.A. unit as the resistance at 0° C. of a wire one metre long weighing one gramme, if the density, 8·95, of cast specimens of their copper be taken as approximately the density of the wires experimented on, which was not determined. To reduce the numbers to accord with the B.A. unit add 1·12 per cent. to the resistances and subtract 1·12 per cent. from the conductivities.

TABLE II.

Cross-section of Round Wires, with Resistance, Conductivity, and Weight of Pure Copper Wires, according to the Birmingham Wire Gauge. (See *Note* to Table I.)

Temperature 15° Cent.

B. W. G.	Diameter.		Area of cross-section.		Resistance.		Conductivity.		Weight. (density = 8·95).	
	Ins	Cms.	Sq. ins.	Sq. cms.	Legal Ohms per Yard.	Legal Ohms per Metre.	Yards per Legal Ohm.	Metres per Legal Ohm.	lbs. per Yard.	Grms. per Metre.
0000	·454	1·153	·162	1·0444	·000150	·000165	6640	6072	1·884	934·7
000	·425	1·079	·142	·915	·000172	·000188	5819	5321	1·651	819·1
00	·380	·965	·113	·732	·000215	·000235	4653	4254	1·320	654·8
0	·340	·864	·0908	·586	·000269	·000294	3735	3415	1·056	524·2
1	·300	·762	·0707	·456	·000345	·000378	2899	2652	·822	408·1
2	·284	·721	·0633	·409	·000385	·000420	2599	2377	·737	365·8
3	·259	·658	·0527	·340	·000463	·000506	2162	1996	·613	304·2
4	·238	·605	·0445	·287	·000548	·000599	1825	1669	·518	256·9
5	·220	·559	·0380	·245	·000642	·000701	1561	1427	·442	219·5
6	·203	·516	·0324	·209	·000754	·000824	1328	1214	·377	186·9
7	·180	·457	·0254	·164	·000958	·00105	1044	1004	·296	146·9
8	·165	·419	·0214	·138	·00114	·00125	877	802	·249	123·5
9	·148	·376	·0172	·111	·00141	·00155	706	645	·200	99·5
10	·134	·340	·0141	·0910	·00173	·00189	578	529	·164	81·4
11	·120	·305	·0113	·0730	·00216	·00235	463	424	·132	65·5
12	·109	·277	·00933	·0602	·00261	·00286	382	350	·109	53·9
13	·095	·241	·00709	·0457	·00344	·00376	291	266	·0825	40·9
14	·083	·211	·00541	·0349	·00451	·00492	221	203	·0630	31·2
15	·072	·183	·00407	·0263	·00599	·00655	167	153	·0474	23·5
16	·065	·165	·00331	·0214	·00735	·00804	136	124	·0386	19·2
17	·058	·147	·00264	·0170	·00923	·0101	108	98·7	·0307	15·3
18	·049	·124	·00189	·0122	·0130	·0141	77·3	70·7	·0220	10·9
19	·042	·107	·00139	·00894	·0176	·0194	56·8	52·0	·0161	8·00
20	·035	·0889	·000962	·00621	·0253	·0277	39·4	36·1	·0122	5·56
21	·032	·0813	·000804	·00519	·0304	·0331	32·0	30·1	·00936	4·64
22	·028	·0711	·000616	·00397	·0395	·0433	25·3	23·1	·00716	3·55
23	·025	·0635	·000491	·00317	·0496	·0543	20·2	18·4	·00571	2·83
24	·022	·0559	·000380	·00245	·0642	·0701	15·6	14·3	·00442	2·19
25	·020	·0508	·000314	·00203	·0778	·0849	12·8	11·7	·00367	1·82
26	·018	·0457	·000254	·00164	·0959	·105	10·2	9·53	·00296	1·47
27	·016	·0406	·000201	·00130	·122	·133	8·25	7·54	·00234	1·16
28	·014	·0356	·000154	·000993	·158	·173	6·31	5·77	·00179	·880
29	·013	·0330	·000133	·000856	·184	·201	5·41	4·98	·00154	·766
30	·012	·0305	·000113	·000732	·216	·235	4·64	4·24	·00132	·653
31	·010	·0254	·0000785	·000507	·311	·339	3·23	2·95	·000915	·454
32	·009	·0229	·0000636	·000410	·384	·419	2·51	2·39	·000746	·367
33	·008	·0203	·0000503	·000324	·486	·530	2·06	1·88	·000585	·290
34	·007	·0178	·0000385	·000248	·634	·693	1·58	1·45	·000442	·220
35	·005	·0127	·0000196	·000127	1·25	1·35	·806	·736	·000229	·113
36	·004	·0102	·0000126	·0000811	1·94	2·13	·516	·471	·000146	·073

TABLE III.

Conductivities of Pure Metals at t° C.*

Conductivity at 0° = 1.

Metal.	Conductivity at t° C.
Silver	$1 - {\cdot}0038278t + {\cdot}0000009848t^2$
Copper	$1 - {\cdot}0038701t + {\cdot}0000009009t^2$
Gold	$1 - {\cdot}0036745t + {\cdot}0000008413t^2$
Zinc	$1 - {\cdot}0037047t + {\cdot}0000008274t^2$
Cadmium	$1 - {\cdot}0036871t + {\cdot}0000007575t^2$
Tin	$1 - {\cdot}0036029t + {\cdot}0000006136t^2$
Lead	$1 - {\cdot}0038756t + {\cdot}0000009146t^2$
Arsenic	$1 - {\cdot}0038996t + {\cdot}0000008879t^2$
Antimony	$1 - {\cdot}0030826t + {\cdot}0000010364t^2$
Bismuth	$1 - {\cdot}0035216t + {\cdot}0000005728t^2$
Iron	$1 - {\cdot}0051182t + {\cdot}0000012916t^2$

* From the results of Matthiessen's experiments; and to be used only for temperatures between 0° C. and 100° C. The formulas, excluding that for iron, agree closely, and give the mean formula $1 - {\cdot}0037647t + {\cdot}00008340t^2$.

TABLE IV.

Conductivity and Resistance of Pure Copper at Temperatures from 0° C. to 40° C.

Calculated by the formula for the Conductivity of Copper in Table III.

Temp.	Conductivity.	Resistance.	Temp.	Conductivity.	Resistance.
0°	1·0000	1·0000	21°	0·9227	1·0838
1	0·9961	1·00388	22	0·9192	1·0879
2	0·9923	1·00776	23	0·9158	1·0920
3	0·9885	1·0116	24	0·9123	1·0961
4	0·9847	1·0156	25	0·9089	1·1003
5	0·9809	1·0195	26	0·9054	1·1044
6	0·9771	1·0234	27	0·9020	1·8085
7	0·9734	1·0274	28	0·8987	1·1127
8	0·9696	1·0313	29	0·8953	1·1169
9	0·9559	1·0353	30	0·8920	1·1211
10	0·9622	1·0393	31	0·8887	1·1253
11	0·9585	1·0433	32	0·8854	1·1295
12	0·9549	1·0473	33	0·8821	1·1337
13	0·9512	1·0513	34	0·8788	1·1379
14	0·9476	1·0553	35	0·8756	1·1421
15	0·9440	1·0593	36	0·8723	1·1461
16	0·9404	1·0634	37	0·8691	1·1506
17	0·9368	1·0675	38	0·8659	1·1548
18	0·9333	1·0715	39	0·8628	1·1591
19	0·9297	1·0755	40	0·8596	1·1633
20	0·9262	1·0797			

Specific Resistances in Legal Ohms of Wires of Different Metals and Alloys.*

Substance.	Resistance at 0° C. of wire one cm. long, one sq. cm. in section.	Resistance at 0° C. of a wire one foot long weighing one grain.	Resistance at 0° C. of a wire one metre long weighing one gramme.	Resistance at 0° C. of a wire one foot long, 1 – 1000th in. in diam.	Resistance at 0° C. of a wire one metre long, one millimetre in diam.	Percentage increase of resistance for 1° C. increase of temperature at 20° C.
Silver, annealed	$1·504 \times 10^{-6}$	0·01916	0·1527	9·049	·2190	0·377
Silver, hard drawn	1·534 ,,	0·02080	0·1661	9·825	·2388	...
Copper, annealed	1·598 ,,	0·02034	0·1424	9·609	·2041	0·388
Copper, hard drawn	1·634 ,,	0·02081	0·1453	9·829	·2083	...
Gold, annealed	2·058 ,,	0·02621	0·4035	12·38	·5784	0·365
Gold, hard drawn	2·095 ,,	0·02667	0·4104	12·60	·5883	...
Aluminium, annealed	2·912 ,,	0·03710	0·0749	17·52	·1073	...
Zinc, pressed	5·614 ,,	9·07163	0·4022	33·84	·576[illegible]	0·365
Platinum, annealed	9·055 ,,	0·1153	1·94	54·47	2·779	...
Iron, annealed	9·715 ,,	0·1237	0·7570	58·44	1·085	...
Nickel, annealed	12·46 ,,	0·1586	1·059	74·93	1·518	...
Tin, pressed	13·21 ,,	0·1682	0·9629	79·45	1·381	0·365
Lead, pressed	19·63 ,,	0·2498	2·232	118·05	3·200	0·387
Antimony, pressed	35·50 ,,	0·4520	2·384	214	3·418	0·389
Bismuth, pressed	131·2 ,,	1·670	12·89	789	18·44	0·354
Mercury, liquid (see *Note*)	95·11 ,,	1·2112	12·92	572·1	18·51	0·072
Platinum - Silver Alloy,† hard or annealed	24·39 ,,	0·3105	2·926	146·69	4·195	0·031
German Silver Alloy, hard or annealed	20·99 ,,	0·2665	1·83	125·89	2·623	0·044
Gold-Silver Alloy,‡ hard or annealed	10·87 ,,	0·1384	1·650	65·36	2·365	0·065
Platinoid *	32·8 ,,	...	...	...	...	021*
Hadfield's manganese steel *	68· about ,,	...	...	...	...	·122*

* Reduced (with the exception of platinoid and manganese steel) from a table given by Professor Jenkin as expressing the results of Matthiessen's experiments. The numbers for platinoid and Hadfield's manganese steel are taken from a paper by Professor J. A. Fleming (*Electrician*, March 9, 1888). The percentage variation of resistance for these two substances is the average for the range between 0° C. and 100° C.

† Two parts platinum, one part silver, by weight.

‡ Two parts gold, one part silver, by weight.

Note.—According to the determinations of the specific resistance of mercury referred to above (p. 243), the value given in this table is about ·8 per cent. too high. A column of pure mercury therefore one sq. millimetre in section, which at 0° C. has a resistance of one ohm, has, according to the B.A. determination of the ohm, a length of ·1048 cms. nearly, and according to Lord Rayleigh and Mrs. Sidgwick's experiments, 106·21 cms. According to the legal ohm the length is, as stated above, p. 72, 106 cms.

The value given in Col. I. for hard-drawn copper has evidently been calculated from the corresponding observed result in Col. III., by using the density 8·89 for copper, and is therefore higher than that on which Tables I. and II. are founded. (See *Note* to Table I.)

TABLE VI.

For reduction of Period of Oscillation observed for finite amplitude to Period for infinitely small amplitude. If T be the observed period and $1 - k$ the reducing factor, so that kT is to be subtracted, the values of k are as follows:—

Amplitude.	k	Amplitude.	k
0	·00000	11	·00230
1	·00002	12	·00274
2	·00008	13	·00322
3	·00017	14	·00373
4	·00030	15	·00428
5	·00048	16	·00487
6	·00069	17	·00550
7	·00093	18	·00616
8	·00122	19	·00686
9	·00154	20	·00761
10	·00190		

TABLE VII.

Units of Work or Energy.

1 erg	$2{\cdot}374 \times 10^{-6}$ foot-poundal.
,, ...	$7{\cdot}375 \times 10^{-8}$ foot-pound at London.
1 centimetre-gramme at Paris . .	981 ergs.
,, ,, ,, at London .	981·17 ergs.
,, ,, ,, ,, ,, .	$2{\cdot}329 \times 10^{-3}$ foot-poundal.
1 metre-kilogramme at Paris . .	981×10^5 ergs
,, ,, ,, at London .	$981{\cdot}17 \times 10^5$ ergs.
,, ,, ,, ...	7·236 foot-pounds.
1 foot-poundal	421390 ergs.
1 foot-pound at London	$13{\cdot}56 \times 10^6$ ergs.
1 *Joule*	10^7 ergs.

TABLE VIII.

Units of Activity or Rate of Working.

1 erg per second	$1{\cdot}34 \times 10^{-10}$ horse power at London.
1 horse power	33000 foot-pounds per minute.
,, ,, at London	$7{\cdot}46 \times 10^9$ ergs per second.
1 force-de-cheval at Paris	$7{\cdot}36 \times 10^9$ ergs per second.
1 *Watt*	10^7 ergs per second.
1 kilowatt	1000 watts.

	0	100	200	300	400	500	600	700	800	900	
0	0.000	20.00	28.28	34.64	40.00	44.72	48.99	52.92	56.57	60.00	0
1	2.000	20.10	28.35	34.70	40.05	44.77	49.03	52.95	56.60	60.03	1
2	2.828	20.20	28.43	34.76	40.10	44.81	49.07	52.99	56.64	60.07	2
3	3.464	20.30	28.50	34.81	40.15	44.86	49.11	53.03	56.67	60.10	3
4	4.000	20.40	28.57	34.87	40.20	44.90	49.15	53.07	56.71	60.13	4
5	4.472	20.49	28.64	34.93	40.25	44.94	49.19	53.10	56.75	60.17	5
6	4.899	20.59	28.71	34.99	40.30	44.99	49.23	53.14	56.78	60.20	6
7	5.292	20.69	28.77	35.04	40.35	45.03	49.27	53.18	56.82	60.23	7
8	5.657	20.78	28.84	35.10	40.40	45.08	49.32	53.22	56.85	60.27	8
9	6.000	20.88	28.91	35.16	40.45	45.12	49.36	53.25	56.89	60.30	9
10	6.325	20.98	28.98	35.21	40.50	45.17	49.40	53.29	56.92	60.33	10
11	6.633	21.07	29.05	35.27	40.55	45.21	49.44	53.33	56.96	60.37	11
12	6.928	21.17	29.12	35.33	40.60	45.25	49.48	53.37	56.99	60.40	12
13	7.211	21.26	29.19	35.38	40.64	45.30	49.52	53.40	57.03	60.43	13
14	7.483	21.35	29.26	35.44	40.69	45.34	49.56	53.44	57.06	60.46	14
15	7.746	21.45	29.33	35.50	40.74	45.39	49.60	53.48	57.10	60.50	15
16	8.000	21.54	29.39	35.55	40.79	45.43	49.64	53.52	57.13	60.53	16
17	8.246	21.63	29.46	35.61	40.84	45.48	49.68	53.55	57.17	60.56	17
18	8.485	21.73	29.53	35.67	40.89	45.52	49.72	53.59	57.20	60.60	18
19	8.718	21.82	29.60	35.72	40.94	45.56	49.76	53.63	57.24	60.63	19
20	8.944	21.91	29.66	35.78	40.99	45.61	49.80	53.67	57.27	60.66	20
21	9.165	22.00	29.73	35.83	41.04	45.65	49.84	53.70	57.31	60.70	21
22	9.381	22.09	29.80	35.89	41.09	45.69	49.88	53.74	57.34	60.73	22
23	9.592	22.18	29.87	35.94	41.13	45.74	49.92	53.78	57.38	60.76	23
24	9.798	22.27	29.93	36.00	41.18	45.78	49.96	53.81	57.41	60.79	24
25	10.000	22.36	30.00	36.06	41.23	45.83	50.00	53.85	57.45	60.83	25
26	10.198	22.45	30.07	36.11	41.28	45.87	50.04	53.89	57.48	60.86	26
27	10.392	22.54	30.13	36.17	41.33	45.91	50.08	53.93	57.52	60.89	27
28	10.583	22.63	30.20	36.22	41.38	45.96	50.12	53.96	57.55	60.93	28
29	10.770	22.72	30.27	36.28	41.42	46.00	50.16	54.00	57.58	60.96	29
30	10.954	22.80	30.33	36.33	41.47	46.04	50.20	54.04	57.62	60.99	30
31	11.136	22.89	30.40	36.39	41.52	46.09	50.24	54.07	57.65	61.02	31
32	11.314	22.98	30.46	36.44	41.57	46.13	50.28	54.11	57.69	61.06	32
33	11.489	23.07	30.53	36.50	41.62	46.17	50.32	54.15	57.72	61.09	33
34	11.662	23.15	30.59	36.55	41.67	46.22	50.36	54.18	57.76	61.12	34
35	11.832	23.24	30.66	36.61	41.71	46.26	50.40	54.22	57.79	61.16	35
36	12.000	23.32	30.72	36.66	41.76	46.30	50.44	54.26	57.83	61.19	36
37	12.166	23.41	30.79	36.72	41.81	46.35	50.48	54.30	57.86	61.22	37
38	12.329	23.49	30.85	36.77	41.86	46.39	50.52	54.33	57.90	61.25	38
39	12.490	23.58	30.92	36.82	41.90	46.43	50.56	54.37	57.93	61.29	39
40	12.649	23.66	30.98	36.88	41.95	46.48	50.60	54.41	57.97	61.32	40
41	12.806	23.75	31.05	36.93	42.00	46.52	50.64	54.44	58.00	61.35	41
42	12.961	23.83	31.11	36.99	42.05	46.56	50.68	54.48	58.03	61.38	42
43	13.115	23.92	31.18	37.04	42.10	46.60	50.71	54.52	58.07	61.42	43
44	13.266	24.00	31.24	37.09	42.14	46.65	50.75	54.55	58.10	61.45	44
45	13.416	24.08	31.30	37.15	42.19	46.69	50.79	54.59	58.14	61.48	45
46	13.565	24.17	31.37	37.20	42.24	46.73	50.83	54.63	58.17	61.51	46
47	13.711	24.25	31.43	37.26	42.28	46.78	50.87	54.66	58.21	61.55	47
48	13.856	24.33	31.50	37.31	42.33	46.82	50.91	54.70	58.24	61.58	48
49	14.000	24.41	31.56	37.36	42.38	46.86	50.95	54.74	58.28	61.61	49
50	14.142	24.49	31.62	37.42	42.43	46.90	50.99	54.77	58.31	61.64	50

	0	100	200	300	400	500	600	700	800	900	
51	14·283	24·58	31·69	37·47	42·47	46·95	51·03	54·81	58·34	61·68	51
52	14·422	24·66	31·75	37·52	42·52	46·99	51·07	54·85	58·38	61·71	52
53	14·560	24·74	31·81	37·58	42·57	47·03	51·11	54·88	58·41	61·74	53
54	14·697	24·82	31·87	37·63	42·61	47·07	51·15	54·92	58·45	61·77	54
55	14·832	24·90	31·94	37·68	42·66	47·12	51·19	54·95	58·48	61·81	55
56	14·967	24·98	32·00	37·74	42·71	47·16	51·22	54·99	58·51	61·84	56
57	15·100	25·06	32·06	37·79	42·76	47·20	51·26	55·03	58·55	61·87	57
58	15·232	25·14	32·12	37·84	42·80	47·24	51·30	55·06	58·58	61·90	58
59	15·362	25·22	32·19	37·89	42·85	47·29	51·34	55·10	58·62	61·94	59
60	15·492	25·30	32·25	37·95	42·90	47·33	51·38	55·14	58·65	61·97	60
61	15·620	25·38	32·31	38·00	42·94	47·37	51·42	55·17	58·69	62·00	61
62	15·748	25·46	32·37	38·05	42·99	47·41	51·46	55·21	58·72	62·03	62
63	15·875	25·53	32·43	38·11	43·03	47·46	51·50	55·24	58·75	62·06	63
64	16·000	25·61	32·50	38·16	43·08	47·50	51·54	55·28	58·79	62·10	64
65	16·125	25·69	32·56	38·21	43·13	47·54	51·58	55·32	58·82	62·13	65
66	16·248	25·77	32·62	38·26	43·17	47·58	51·61	55·35	58·86	62·16	66
67	16·371	25·85	32·68	38·31	43·22	47·62	51·65	55·39	58·89	62·19	67
68	16·492	25·92	32·74	38·37	43·27	47·67	51·69	55·43	58·92	62·23	68
69	16·613	26·00	32·80	38·42	43·31	47·71	51·73	55·46	58·96	62·26	69
70	16·733	26·08	32·86	38·47	43·36	47·75	51·77	55·50	58·99	62·29	70
71	16·852	26·15	32·92	38·52	43·41	47·79	51·81	55·53	59·03	62·32	71
72	16·971	26·23	32·98	38·57	43·45	47·83	51·85	55·57	59·06	62·35	72
73	17·088	26·31	33·05	38·63	43·50	47·87	51·88	55·61	59·09	62·39	73
74	17·205	26·38	33·11	38·68	43·54	47·92	51·92	55·64	59·13	62·42	74
75	17·321	26·46	33·17	38·73	43·59	47·96	51·96	55·68	59·16	62·45	75
76	17·436	26·53	33·23	38·78	43·63	48·00	52·00	55·71	59·19	62·48	76
77	17·550	26·61	33·29	38·83	43·68	48·04	52·04	55·75	59·23	62·51	77
78	17·664	26·68	33·35	38·88	43·73	48·08	52·08	55·79	59·26	62·55	78
79	17·776	26·76	33·41	38·94	43·77	48·12	52·12	55·82	59·30	62·58	79
80	17·889	26·83	33·47	38·99	43·82	48·17	52·15	55·86	59·33	62·61	80
81	18·000	26·91	33·53	39·04	43·86	48·21	52·19	55·89	59·36	62·64	81
82	18·111	26·98	33·59	39·09	43·91	48·25	52·23	55·93	59·40	62·67	82
83	18·221	27·06	33·65	39·14	43·95	48·29	52·27	55·96	59·43	62·71	83
84	18·330	27·13	33·70	39·19	44·00	48·33	52·31	56·00	59·46	62·74	84
85	18·439	27·20	33·76	39·24	44·05	48·37	52·35	56·04	59·50	62·77	85
86	18·547	27·28	33·82	39·29	44·09	48·41	52·38	56·07	59·53	62·80	86
87	18·655	27·35	35·88	39·34	44·14	48·46	52·42	56·11	59·57	62·83	87
88	18·762	27·42	33·94	39·40	44·18	48·50	52·46	56·14	59·60	62·86	88
89	18·868	27·50	34·00	39·45	44·23	48·54	52·50	56·18	59·63	62·90	89
90	18·974	27·57	34·06	39·50	44·27	48·58	52·54	56·21	59·67	62·93	90
91	19·079	27·64	34·12	39·55	44·32	48·62	52·57	56·25	59·70	62·96	91
92	19·183	27·71	34·18	39·60	44·36	48·66	52·61	56·28	59·73	62·99	92
93	19·287	27·78	34·23	39·65	44·41	48·70	52·65	56·32	59·77	63·02	93
94	19·391	27·86	34·29	39·70	44·45	48·74	52·69	56·36	59·80	63·06	94
95	19·494	27·93	34·35	39·75	44·50	48·79	52·73	56·39	59·83	63·09	95
96	19·596	28·00	34·41	39·80	44·54	48·83	52·76	56·43	59·87	63·12	96
97	19·698	28·07	34·47	39·85	44·59	48·87	52·80	56·46	59·90	63·15	97
98	19·799	28·14	34·53	39·90	44·63	48·91	52·84	56·50	59·93	63·18	98
99	19·900	28·21	34·58	39·95	44·68	48·95	52·88	56·53	59·97	63·21	99
100	20·000	28·28	34·64	40·00	44·72	48·99	52·92	56·57	60·00	63·25	100

TABLE X.

French and English Units.

Metre in inches	39·37043
Foot in centimetres	30·4797
Mile in kilometres	1·6093
Kilomètre in miles	·62137
Gramme in grains	15·43235
Grain in milligrammes	64·799
Pound in grammes	453·593
Kilogramme in pounds	2·2046
British ton in French tons of 1000 kilos	1·016
Litre in cubic inches	61·0253
Cubic inch in cubic centimetres	16·3862
Cubic foot in cubic centimetres	28315·3

Virtual Resistance of Conductors carrying Alternating Currents.

Sir William Thomson has given (Presidential Address Inst. Elect. Engs. Jan. 9, 1889) the following formulæ for calculating the resistance of wires carrying alternating currents.

Let σ denote the specific resistance of the wire in C.G.S. units (see above, p. 238).

a ,, the radius of the wire.

$R(S)$,, the value of $\sigma l/\pi a^2$, that is the resistance in cms. per second of a length of l cms. of the wire with a steady current flowing through it.

$R(N)$,, the effective ohmic * resistance of the same length, l, with an alternate current of N periods per second flowing through it.

$c(N)$,, the current density at distance r from the axis at time t.

$C(N)$,, the current density at the axis at time t.

then $$c(N) = C(N)(\text{ber}\, q \cos\theta - \text{bei}\, q \sin\theta) \quad . \; . \; . \; . \; . \; . \quad (1)$$

where $q = 2\pi r\sqrt{2N/\sigma}$, $\theta = 2\pi Nt$, and ber and bei denote two functions defined as follows

$$\text{ber}\, q = 1 - \frac{q^4}{2^2 . 4^2} + \frac{q^8}{2^2 . 4^2 . 6^2 . 8^2} - \&\text{c.}$$

$$\text{bei}\, q = \frac{q^2}{2^2} - \frac{q^6}{2^2 . 4^2 . 6^2} + \&\text{c.}$$

Also if p denote the value of q with $r = a$,

$$\frac{R(N)}{R(S)} = \tfrac{1}{2} p \frac{\text{ber}\, p . \text{bei}' p - \text{bei}\, p\, \text{ber}' p}{(\text{ber}' p)^2 + (\text{bei}' p)^2} \quad . \; . \; . \; . \quad (2)$$

where the accents denote differential coefficients.

For copper at 0° C. $\sigma = 1610$ sq. cms. per sec. Hence with $N = 80$ $q = 2r$ nearly.

* Not including any impedance effect of self-induction.

Thus for copper with $N = 80$, the column below headed q may be taken as containing the diameters of the wires and in respect to the distribution of the current through the wire expressed by the formula in (1) above, q may be taken as the diameter of the cylindric shell in which the current density is to be calculated.

TABLE (XI.) OF NUMERICAL VALUES.

(Calculated for Sir W. Thomson by Mr. M. Maclean.)

q	$\dfrac{\text{bei}'\,\text{ber} - \text{ber}'\,\text{bei}}{\text{ber}'^2 + \text{bei}'^2}$	$\tfrac{1}{2}q\,\dfrac{\text{bei}'\,\text{ber} - \text{ber}'\,\text{bei}}{\text{ber}'^2 + \text{bei}'^2}$
0·0	∞	1·0000
0·5	4·0000	1·0000
1·0	2·0014	1·0001
1·5	1·3678	1·0258
2·0	1·0805	1·0805
2·5	·9398	1·1747
3·0	·8787	1·3180
3·5	·8526	1·4920
4·0	·8389	1·6778
4·5	·8279	1·8628
5·0	·8172	2·0430
5·5	·8069	2·2190
6·0	·7979	2·3937
8	·7739	3·0956
10	·7588	3·7940
15	·7431	5·5732
20	·7325	7·3250

From the data here given and some further data supplied by Sir William Thomson, Mr. W. M. Mordey has calculated the following table :—

(See preceding page.)

TABLE XII.

VIRTUAL RESISTANCE OF CONDUCTORS CARRYING ALERNATING CURRENTS.

From Mr. W. M. Mordey's paper on Alternate Current Working, Inst. El. Eng., May 23, 1889, *Electrician*, May 31.

Diameter.		Area.		Percentage Increase of Ordinary Resistance.	Number of Complete Periods per Second.
Millimetres.	Inches.	Sq. mm.	Sq. in.		
10	·3937	78·54	·122	Less than $\frac{1}{100}$	80
15	·5905	176·7	·274	2·5	
20	·7874	314·16	·487	8	
25	·9842	490·8	·760	17·5	
40	1·575	1,256	1·95	68	
100	3·937	7,854	12·17	3.8 times	
1000	39·39	785,400	1.217	35 times	
9	·3543	63·62	·098	Less than $\frac{1}{100}$	100
13·4	·5280	141·3	·218	2·5	
18	·7086	254·4	·394	8	
22·4	·8826	394	·611	17·5	
7·75	·3013	47·2	·071	Less than $\frac{1}{100}$	133
11·61	·4570	106	·164	2·5	
15·5	·6102	189	.292	8	
19·36	·7622	294	·456	17·5	

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